NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 5

Question 1 (i). It is a geometric series with the first term $a = 3/\pi^3$ and common ratio $r = 3/\pi$. Thus, $S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{\frac{3}{\pi^3}(1-(\frac{3}{\pi})^n)}{1-\frac{3}{\pi}}$.

Since $\dfrac{3}{\pi} < 1$, it follows that $\lim_{n\to\infty}(\dfrac{3}{\pi})^n = 0$. Thus, $\lim_{n\to\infty} S_n = \dfrac{3}{\pi^2(\pi-3)}$ (converges). □

Question 1 (ii). Let $a_n = \ln\dfrac{n+2}{n+3}$. Then the partial sum

$$\begin{aligned} S_n = a_1 + a_2 + \cdots + a_n &= \ln\frac{1+2}{1+3} + \ln\frac{2+2}{2+3} + \ln\frac{3+2}{3+3} + \cdots + \ln\frac{n+2}{n+3} \\ &= \ln\frac{(1+2)(2+2)(3+2)(4+2)\cdots(n-1+2)(n+2)}{(1+3)(2+3)(3+3)(4+3)\cdots(n-1+3)(n+3)} \\ &= \ln\frac{3\cdot4\cdot5\cdot6\cdots(n+1)\cdot(n+2)}{4\cdot5\cdot6\cdot7\cdots(n+2)\cdot(n+3)} = \ln\frac{3}{n+3} = \ln 3 - \ln(n+3). \end{aligned}$$

Thus $\{S_n\}$ is divergent and so is the series $\sum_{n=1}^{\infty}\ln\dfrac{n+2}{n+3}$. □

Question 1 (iii). Let $a_n = \dfrac{1}{n(n+2)}$. Observe that

$$\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

The partial sum

$$\begin{aligned} S_n &= \frac{1}{1\cdot(1+2)} + \frac{1}{2\cdot(2+2)} + \cdots + \frac{1}{n\cdot(n+2)} \\ &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n} - \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n+2}\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right). \end{aligned}$$

Thus the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ is convergent and

$$\sum_{n=1}^{\infty}\frac{1}{n(n+2)}=\lim_{n\to\infty}S_n=\frac{1}{2}\cdot\frac{3}{2}=\frac{3}{4}.$$

□

Question 2 (i). $\displaystyle\lim_{n\to\infty}a_n=-\frac{1}{2}\neq 0$. Thus by the n-th term test, $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges.

□

Question 2(ii). A simple check gives

$$\{a_n\}=\{-1,0,-1,2,-1,0,-1,2,-1,0,-1,2,\cdots\}.$$

(Note that $a_{n+4}=a_n$ for all n, i.e. this sequence has a repeating pattern which repeats when n increases by 4). Thus, $\displaystyle\lim_{n\to\infty}a_n$ does not exist since it has two convergent subsequences which converge to two different limits (e.g. $\displaystyle\lim_{k\to\infty}a_{2k-1}=-1$ and $\displaystyle\lim_{k\to\infty}a_{4k-2}=0$.). Thus by the n-th term test, $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges.

□

Question 3(i). Diverges, use LCT, vs $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.

(Rough work: $\displaystyle a_n=\frac{n^2+\sqrt{n}}{n^3-n+3}\sim\frac{n^2}{n^3}=\frac{1}{n}$.)

Here is what you should actually write in your solution:

Solution:

$\displaystyle\frac{n^2+\sqrt{n}}{n^3-n+3}\geq 0,\quad \frac{1}{n}\geq 0\quad\forall n\geq 1.$

$\displaystyle\lim_{n\to\infty}\frac{\frac{n^2+\sqrt{n}}{n^3-n+3}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{1+\frac{1}{n^{3/2}}}{1-\frac{1}{n^2}+\frac{1}{n^3}}=1(\neq 0,\ \neq\infty).$

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent (since it is a p-series with $p=1\leq 1$).

Thus by the limit comparison test, $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+\sqrt{n}}{n^3-n+3}$ also diverges. □

Question 3(ii). Converges, use CT, vs $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^3}$.

(Rough work: $\displaystyle -1\leq\sin n\leq 1\implies 2\leq 3+\sin n\leq 4\implies\frac{2}{n^3}\leq\frac{3+\sin n}{n^3}\leq\frac{4}{n^3}$.

But the inequality $\displaystyle\frac{2}{n^3}\leq\frac{3+\sin n}{n^3}$ is useless, since it basically only says that the given series is bigger than a convergent series, and thus the given series may still be convergent or divergent. Thus we use the other inequality $\dfrac{3+\sin n}{n^3} \le \dfrac{4}{n^3}$, which amounts to comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^3}$.) Solution: For all $n \ge 1$,

$$\begin{aligned} -1 &\le \sin n \le 1 \\ \implies 2 &\le 3+\sin n \le 4 \\ \implies 0 &\le \frac{2}{n^3} \le \frac{3+\sin n}{n^3} \le \frac{4}{n^3}. \end{aligned}$$

Thus, $a_n = \dfrac{3+\sin n}{n^3} \ge 0$ for all $n \ge 1$. Also, the series $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^3} = 4\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, since it is the p-series with $p = 3 > 1$. Thus, by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{3+\sin n}{n^3}$ converges. □

Question 3(iii). Converges, use LCT, vs $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}}$.

Rough work: $a_n = \dfrac{2^n+3}{3^{n+1}-n} \sim \dfrac{2^n}{3^{n+1}}$.

Solution: For all $n \ge 1$, $a_n = a_n = \dfrac{2^n+3}{3^{n+1}-n} \ge 0$.

$$\lim_{n\to\infty} \frac{\frac{2^n+3}{3^{n+1}-n}}{\frac{2^n}{3^{n+1}}} = \lim_{n\to\infty} \frac{1+\frac{3}{2^n}}{1-\frac{n}{3^{n+1}}} = \frac{1+0}{1-0} = 1 \quad (\ne 0, \ne \infty).$$

Now, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}}$ is convergent since it is a geometric series with common ration $\dfrac{2}{3} < 1$. Thus, by the LCT, $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3}{3^{n+1}-n}$ converges. □

Question 3(iv). diverges, use CT, vs $\displaystyle\sum_{n=1}^{\infty} \frac{3}{6n}$.

Rough work: $-1 \le (-1)^n \le 1 \implies 3 \le 4+(-1)^n \le 5 \implies \dfrac{3}{6n} \le \dfrac{4+(-1)^n}{6n} \le \dfrac{5}{6n}$.

Note that the inequality $\dfrac{4+(-1)^n}{6n} \le \dfrac{5}{6n}$ is useless, since it just says that the given series is smaller than a divergent series. On the other hand, the other inequality $\dfrac{3}{6n} \le \dfrac{4+(-1)^n}{6n}$ is useful since it basically says that the given series is bigger than a divergent series.

Solution: For all $n \ge 1$,

$$-1 \le (-1)^n \le 1$$

$$\Longrightarrow 3 \le 4 + (-1)^n \le 5$$
$$\Longrightarrow 0 \le \frac{3}{6n} \le \frac{4 + (-1)^n}{6n} \le \frac{5}{6n}.$$

In particular, $a_n \ge 0$ for all $n \ge 1$. Also, $\sum_{n=1}^{\infty} \frac{3}{6n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, since it is the p-series with $p = 1 \le 1$. Thus, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{4+(-1)^n}{6n}$ diverges. □

Question 3 (v). Diverges, use LCT, vs $\sum_{n=1}^{\infty} \frac{1}{n}$ or $\sum_{n=1}^{\infty} \frac{2}{n}$.

Let $a_n = \frac{2}{n^{1+\frac{1}{n}}}$ and let $b_n = \frac{1}{n}$. Observe that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{n}}}{2} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\sum_{n=1}^{\infty} \frac{2}{n^{1+\frac{1}{n}}}$ is divergent by the limit comparison test. □

Question 4(a). Since $\sum_{n=1}^{\infty} a_n$ is convergent, we have $\lim_{n\to\infty} a_n = 0$ and so there exists a positive integer N such that $a_n = |a_n| = |a_n - 0| < 1$ for $n > N$. Since $a_n \ge 0$,

$$a_n^2 = a_n \cdot a_n \le 1 \cdot a_n = a_n$$

for $n > N$. By the comparison test, the positive series $\sum_{n=1}^{\infty} a_n^2$ is convergent. □

Another Solution of Question 4 (a). Since $\sum_{n=1}^{\infty} a_n$ is convergent, we have $\lim_{n\to\infty} a_n = 0$. Let $b_n = a_n^2$. Then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} a_n = 0,$$

that is, $b_n << a_n$. Since $a_n \ge 0$, by limit comparison test, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n^2$ is convergent. □

Question 2 (b). Let $a_n = \frac{1}{n^2}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} \sqrt{a_n}$ is divergent by the p-series. □