

NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I

MA2108

Advanced Calculus II

Solutions to Tutorial 6

Question 1 (a). Let $f(x) = \frac{1}{x(1 + \ln x)}$. Then $f(x)$ is a positive monotone decreasing function over $[1, +\infty)$. Since

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x(1 + \ln x)}dx \stackrel{y=\ln x}{=} \int_0^\infty \frac{1}{1 + y}dy = \ln(1 + y)\Big|_0^\infty = +\infty.$$

is divergent, the series $\sum_{n=1}^\infty \frac{1}{n(1 + \ln n)}$ is divergent by the integral test. \square

Question 2 (b). Let $f(x) = \frac{1}{x[1 + (\ln x)^2]}$. Then $f(x)$ is a positive monotone decreasing function over $[1, +\infty)$. Since

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x[1 + (\ln x)^2]}dx \stackrel{y=\ln x}{=} \int_0^\infty \frac{1}{1 + y^2}dy = \arctan y\Big|_0^\infty = \frac{\pi}{2}$$

is convergent, the series $\sum_{n=1}^\infty \frac{1}{n(1 + \ln n)}$ is convergent by the integral test. \square

Question 2 (a). Let $a_n = \frac{(3n)!}{6^n n!(2n)!}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{[3(n+1)]!6^n n!(2n)!}{6^{n+1}(n+1)![2(n+1)]!(3n)!} \\ &= \lim_{n \rightarrow \infty} \frac{(3n+3)(3n+2)(3n+1)}{6(n+1)(2n+2)(2n+1)} = \frac{3 \cdot 3 \cdot 3}{6 \cdot 2 \cdot 2} = \frac{27}{24} > 1. \end{aligned}$$

Thus the series $\sum_{n=1}^\infty \frac{(3n)!}{6^n n!(2n)!}$ is divergent by the ratio test. \square

Question 2 (b). Let $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{n}\right)^n = \frac{2}{e} < 1$, the series $\sum_{n=1}^\infty a_n$ is convergent by the ratio test. \square

Question 3 (a). Let $a_n = \frac{5n^2 \cdot 3^n}{4^{n+4}}$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{5^{\frac{1}{n}}(\sqrt[n]{n})^2 \cdot 3}{4 \cdot 4^{\frac{4}{n}}} = \frac{1 \cdot 1^2 \cdot 3}{4 \cdot 1} = \frac{3}{4} < 1.$$

Thus the series $\sum_{n=1}^{\infty} \frac{5n^2 \cdot 3^n}{4^{n+4}}$ is convergent by the simplified root test. \square

Question 3 (b). Let $a_n = \frac{3^{2n}}{5^n} \left(1 - \frac{1}{2n}\right)^{n^2}$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{3^2}{5} \left(1 + \frac{-\frac{1}{2}}{n}\right)^n = \frac{9}{5} e^{-\frac{1}{2}} = \frac{9}{5\sqrt{e}} > 1$$

because $e < \frac{9^2}{5^2} = \frac{81}{25} = 3.24$. Thus the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n} \left(1 - \frac{1}{2n}\right)^{n^2}$ is divergent by the simplified root test. \square

Question 3 (c). Let a_n be the n -term in the series. Then $a_{2n-1} = \frac{1}{4^{2n-1}}$ and $a_{2n} = \frac{1}{5^{2n}}$. Thus

$$\sqrt[n]{a_n} = \begin{cases} \frac{1}{4} & \text{if } n \text{ is odd} \\ \frac{1}{5} & \text{if } n \text{ is even} \end{cases}$$

and so $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{4} < 1$. Hence the series is convergent by the root test. \square

Question 4. (i). Converges, simplified root test, $\ell = \frac{1}{e^2} < 1$.

(ii) converges, LCT, vs $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(iii) converges, LCT, vs $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ (remark: In fact, one may compare with any p -series with $1 < p < 1.2$).

Rough work: For $p > 0$, recall that

$$1 \leq \ln n \leq n^p \quad \forall n \text{ large}$$

$$\frac{1}{n^{1.2}} \leq \frac{\ln n}{n^{1.2}} \leq \frac{n^p}{n^{1.2}} = \frac{1}{n^{1.2-p}} \quad \forall n \text{ large}$$

Although the inequality on the left, i.e. $\frac{1}{n^{1.2}} \leq \frac{\ln n}{n^{1.2}}$, is correct, it is useless since it only says that the given series $\sum_{n=1}^{\infty} a_n$ is bigger than a convergent series $\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$. The

series $\sum_{n=1}^{\infty} a_n$ may still be convergent or divergent.

To make use of the inequality on the right, i.e. $\frac{\ln n}{n^{1.2}} \leq \frac{1}{n^{1.2-p}}$, one observes that the only way to get a conclusion is that the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.2-p}}$ is already convergent (so

that the given series is smaller than a convergent series). This means that we must have $1.2 - p > 1 \implies p < 0.2$. Hence we should choose $0 < p < 0.2$. As it turns out, any p with $0 < p < 0.2$ (corresponding to comparing with P -series with $1 < P < 1.2$) will yield the desired conclusion.

(iv) diverges, ratio test, $\ell = \frac{6}{5} > 1$.

(v) diverges, integral test.

(vi) diverges, LCT, vs $\sum_{n=1}^{\infty} \frac{1}{n}$.

(Remark: One may compare the given series with any p -series with $\frac{1}{2} < p \leq 1$).
Rough work is similar to (iii).

□