NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 7

Question 1. (a) Let $a_n = \dfrac{1}{\sqrt{n}(1+\ln n)^2}$. Then

(i) $a_n = \dfrac{1}{\sqrt{n}(1+\ln n)^2} > 0 \quad \forall n \geq 2$.

(ii) Since both $\sqrt{n}$ and $(1+\ln n)^2$ increases with n, it follows that $\{a_n\}$ is monotone decreasing.

(iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{\sqrt{n}(1+\ln n)^2} = 0$.

Therefore, by the alternating series test, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}(1+\ln n)^2}$ converges.

(b) By the Alternate Series Estimation,

$$\begin{aligned} R_{100} &= \left|S - \sum_{n=2}^{100} \frac{(-1)^{n+1}}{\sqrt{n}(1+\ln n)^2}\right| \\ &\leq a_{101} = \frac{1}{\sqrt{101}(1+\ln 101)^2} \approx 3.16 \times 10^{-3}. \end{aligned}$$

□

Question 2 (i). Let $a_n = \dfrac{\ln n}{\sqrt{n}}$. Then $a_n \geq 0$. We show that a_n is eventually monotone decreasing. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \frac{1}{2\sqrt{x}}}{x} = \frac{2-\ln x}{2x^{\frac{3}{2}}} \leq 0$$

for $x \geq e^2$ and so $\{a_n\}$ is monotone decreasing for $n \geq 9$. Since $\lim_{n\to\infty} a_n = 0$, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{\sqrt{n}}$ is convergent by the alternating series test. □

Question 2 (ii). Since $\left|(-1)^{n+1}\dfrac{\ln n}{\sqrt{n}}\right| \geq \dfrac{1}{n^{\frac{1}{2}}}$ for $n \geq 3$ and the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent by the p-series, the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{\ln n}{\sqrt{n}}\right|$ is divergent by the comparison test. By (i), the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{\sqrt{n}}$ is conditionally convergent. □

Question 3. Let $a_n = (-1)^n \dfrac{\cos n}{2^n}$. Then $|a_n| \leq \dfrac{1}{2^n}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent by the geometric series, the series $\displaystyle\sum_{n=1}^{\infty} \left|(-1)^n \frac{\cos n}{2^n}\right|$ is convergent by the comparison test and so the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{2^n}$ is absolutely convergent. □

Question 4 (a). This series is conditionally convergent because it is convergent by the alternating series test and the series $\displaystyle\sum_{n=1}^{\infty} \left|(-1)^n \frac{3}{2n+1}\right|$ is divergent by the limit comparison test with the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. □

Question 4 (b). Let $a_n = (-1)^n \dfrac{n}{4n+3}$. Then $\displaystyle\lim_{n\to\infty} a_{2n-1} = -\frac{1}{4}$ and $\displaystyle\lim_{n\to\infty} a_{2n} = \frac{1}{4}$. Thus the limit of $(-1)^n \dfrac{n}{4n+3}$ does not exist and so the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{4n+3}$ is divergent by the divergence test. □

Question 4 (c). Let $a_n = (-1)^n \left(\dfrac{1+2n}{3+4n}\right)^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1+2n}{3+4n} = \frac{2}{4} = \frac{1}{2} < 1.$$

Thus the positive series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent by the simplified root test and so the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{1+2n}{3+4n}\right)^n$ is absolutely convergent. □

Question 4 (d). Observe

$$\left|(-1)^{n+1} \frac{\cos n}{n(\ln n)^2}\right| \leq \frac{1}{n(\ln n)^2}.$$

Let $f(x) = \dfrac{1}{x(\ln x)^2}$. Then $f(x)$ is positive and monotone decreasing on $[2, +\infty)$. Since the integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx \xlongequal[dy=\frac{1}{x}dx]{y=\ln x} \int_{\ln 2}^{\infty} \frac{1}{y^2} dy = -\frac{1}{y}\Big|_{\ln 2}^{\infty} = \frac{1}{\ln 2}$$

is convergent, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent by the integral test. By the comparison test, the series

$$\sum_{n=2}^{\infty} \left| (-1)^{n+1} \frac{\cos n}{n(\ln n)^2} \right|$$

is convergent and so the series

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\cos n}{n(\ln n)^2}$$

is absolutely convergent. □

Question 5 (a). $F(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{nx} = (e^x)^x = e^{x^2}$ □

Question 5 (b). It does not converge pointwise because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist. □

Question 5 (c).

$$F(x) = \lim_{n\to\infty} \frac{x^{2n}}{1 + x^{2n}} = \begin{cases} \frac{0}{1+0} = 0 & \text{if} \quad 0 \le x < 1 \\ \frac{1}{2} & \text{if} \quad x = 1 \end{cases}$$

□