NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 8

Question 1 (a). The limiting function $F(x) = \lim_{n\to\infty} \frac{n^2}{n^2+x^2} = \lim_{n\to\infty} \frac{1}{1+\frac{x^2}{n^2}} = 1.$

$$0 \le T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} |\frac{n^2}{n^2+x^2} - 1| = \sup_{0\le x\le 1} \frac{x^2}{n^2+x^2} \le \frac{1}{n^2}.$$

Since $\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so $\{F_n\}$ converges uniformly on $[0, 1]$. □

Question 1 (b). The limiting function $F(x) = \lim_{n\to\infty} x^n(1-x) = 0$ for $0 \le x \le 1$. Observe

$$T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} x^n(1-x).$$

Let $g(x) = x^n(1-x)$. Then $g'(x) = nx^{n-1}(1-x) - x^n = x^{n-1}[n-(n+1)x]$. From $g'(x) = 0$, we have $x = 0$ or $\frac{n}{n+1}$. Since $g'(x) \ge 0$ for $0 \le x \le \frac{n}{n+1}$ and $g'(x) \le 0$ for $\frac{n}{n+1} \le x \le 1$, $\sup_{0\le x\le 1} g(x) = \max\{g(x)|0 \le x \le 1\} = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$ and so

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \left(1-\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0.$$

Thus $\{F_n\}$ converges uniformly on $[0, 1]$. □

Question 1 (c). The limiting function $f(x) = \lim_{n\to\infty} \frac{n \ln x}{x^n} = 0$ for $1 \le x < \infty$. Observe that

$$T_n = \sup_{x\ge 1} |f_n(x) - f(x)| = \sup_{x\ge 1} \left|\frac{n\ln x}{x^n} - 0\right| = \sup_{x\ge 1} \frac{n \ln x}{x^n}.$$

Let $g(x) = \frac{n\ln x}{x^n}$. From

$$g'(x) = \left(n \ln x \cdot x^{-n}\right)' = n\frac{1}{x}x^{-n} - n^2 \ln x \cdot x^{-n-1} = \frac{n - n^2 \ln x}{x^{n+1}} = 0,$$

we have $x = e^{\frac{1}{n}}$. Since $g'(x) \ge 0$ for $1 \le x \le e^{\frac{1}{n}}$ and $g'(x) \le 0$ for $x \ge e^{\frac{1}{n}}$, we have

$$T_n = \sup_{x\ge 1} g(x) = \max\{g(x)|x \ge 1\} = \frac{n \ln e^{\frac{1}{n}}}{\left(e^{\frac{1}{n}}\right)^n} = \frac{1}{e}$$

and so $\lim_{n\to\infty} T_n = \dfrac{1}{e} \neq 0$. Thus $\{f_n\}$ does not converge uniformly on $[1, +\infty)$. $\square$

Question 1 (d). The limiting function $f(x) = \lim_{n\to\infty} \dfrac{n \ln x \cos nx}{x^n} = 0$ for $x \geq 4$. Observe that

$$0 \leq T_n = \sup_{x\geq 4} |f_n(x) - f(x)| = \sup_{x\geq 4} \frac{n \ln x \cdot |\cos nx|}{x^n} \leq \sup_{x\geq 4} \frac{n \ln x}{x^n} = g(4) = \frac{n \ln 4}{4^n},$$

where $g(x) = \dfrac{n \ln x}{x^n}$ is monotone decreasing on $[4, +\infty)$. Since $\lim_{n\to\infty} \dfrac{n \ln 4}{4^n} = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so $\{f_n\}$ converges uniformly on $[4, +\infty)$. $\square$

Question 1 (e). The limiting function $F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \dfrac{n^2}{n^2 + x^2} = 1$. Observe that

$$T_n = \sup_{x\geq 0} |F_n(x) - F(x)| = \sup_{x\geq 0} \left| \frac{n^2}{n^2 + x^2} - 1 \right| = \sup_{x\geq 0} \frac{x^2}{n^2 + x^2}.$$

Let $g(x) = \dfrac{x^2}{n^2 + x^2}$. Since

$$g'(x) = \frac{2x(n^2 + x^2) - x^2 \cdot 2x}{(n^2 + x^2)^2} = \frac{2xn^2}{(n^2 + x^2)^2} \geq 0$$

for $x \geq 0$, the function $g(x)$ is monotone increasing and so

$$T_n = \sup_{x\geq 0} g(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^2}{n^2 + x^2} = \lim_{x\to\infty} \frac{2x}{2x} = 1.$$

Thus $\lim_{n\to\infty} T_n = 1 \neq 0$ and so $\{F_n(x)\}$ does not converge uniformly on $[0, +\infty)$. $\square$

Question 2. (i) Let $\epsilon = 1$. Since $\{F_n\}$ converges uniformly to F on I, there exists $N > 0$ such that

$$|F_n(x) - F(x)| < 1 \quad \forall n > N \text{ and } \forall x \in I.$$

Fix $n = N + 1$. Then

$$|F_{N+1}(x) - F(x)| < 1 \quad \forall x \in I.$$

Also it is given that

$$|F_{N+1}(x)| \leq M_{N+1} \quad \forall x \in I.$$

Hence, for all $x \in I$,

$$\begin{aligned} |F(x)| &= |(F(x) - F_{N+1}(x)) + F_{N+1}(x)| \\ &\leq |(F(x) - F_{N+1}(x))| + |F_{N+1}(x)| \\ &\leq 1 + M_{N+1}. \end{aligned}$$

We may take $M = 1 + M_{N+1}$.

(ii) By (i), there exists $M > 0$ such that $|F(x)| \le M$ for all $x \in I$. Given any $\epsilon > 0$, since $\{F_n\}$ converges uniformly to F on I, there exists $N \in \mathbb{N}$ such taht

$$|F_n(x) - F(x)| \le \frac{\epsilon}{M} \quad \forall x \in I \text{ and } n > N.$$

Then for all $x \in I$ and $n > N$,

$$|F_n(x)F(x) - F(x)F(x)| = |F_n(x) - F(x)| \cdot |F(x)|$$

$$< \frac{\epsilon}{M} \cdot M = \epsilon.$$

Hence $\{F_nF\}$ converges uniformly (to $(F)^2$) on I.

□

Question 3 (i). Since

$$\left|\frac{\cos nx}{n^2 + x^2}\right| \le \frac{1}{n^2}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2}$ converges uniformly on $(-\infty, +\infty)$ by the Weierstrass M-test. □

Question 3 (ii). Since

$$\left|\frac{1}{1 + n^3x^2}\right| \le \frac{1}{1 + 2^2n^3} \le \frac{1}{n^3}$$

for $x \ge 2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty} \frac{1}{1 + n^3x^2}$ converges uniformly on $[2, +\infty)$ by the Weierstrass M-test. □

Question 3 (iii). Let $f_n(x) = \frac{xe^{-x}}{n^2}$ for $x \in (0, +\infty)$. Then

$$f_n'(x) = \frac{1}{n^2}\left(e^{-nx} - xne^{-nx}\right) = \frac{(1 - nx)e^{-nx}}{n^2}.$$

Thus $f_n(x)$ is monotone increasing for $0 \le x \le \frac{1}{n}$ and monotone decreasing for $x \ge \frac{1}{n}$. It follows that

$$|f_n(x)| \le f_n\left(\frac{1}{n}\right) = \frac{\frac{1}{n}e^{-n\cdot\frac{1}{n}}}{n^2} = \frac{e^{-1}}{n^3}$$

for $x \in (0, +\infty)$. Since the series $\sum_{n=1}^{\infty} \frac{e^{-1}}{n^3} = e^{-1} \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty} \frac{xe^{-nx}}{n^2}$ converges uniformly on $(0, +\infty)$ by the Weierstrass M-test. □

Question 4 (i). Let $F_n(x) = \frac{n + e^x}{n + x^2}$. Then the limiting function

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{n + e^x}{n + x^2} = \lim_{n\to\infty} \frac{1 + \frac{e^x}{n}}{1 + \frac{x^2}{n}} = 1$$

and

$$\begin{aligned} 0 \le T_n &= \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} \left| \frac{n + e^x}{n + x^2} - 1 \right| \\ &= \sup_{0\le x\le 1} \frac{|e^x - x^2|}{n^2 + x^2} \le \sup_{0\le x\le 1} \frac{e^x + x^2}{n^2 + x^2} \le \frac{e + 1}{n^2} \end{aligned}$$

Since $\lim_{n\to\infty} \frac{e+1}{n^2} = \lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem. Thus $\{F_n\}$ converges uniformly to $F(x)$ and so

$$\lim_{n\to\infty} \int_0^1 \frac{n + e^x}{n + x^2} dx = \int_0^1 \lim_{n\to\infty} F_n(x) dx = \int_0^1 1 dx = 1.$$

□

Question 4 (ii). Let $F_n(x) = \left(\frac{x^2 + 1}{8}\right)^n \sin nx$. Then

$$0 \le |F_n(x)| = \left| \left(\frac{x^2 + 1}{8}\right)^n \sin nx \right| \le \left(\frac{2^2 + 1}{8}\right)^n = \left(\frac{5}{8}\right)^n$$

for $1 \le x \le 2$. Since $\lim_{n\to\infty} \left(\frac{5}{8}\right)^n = \lim_{n\to\infty} 0 = 0$, the limiting function $F(x) = 0$ for $1 \le x \le 2$ and

$$0 \le T_n = \sup_{1\le x\le 2} |F_n(x) - F(x)| = \sup_{1\le x\le 2} |F_n(x)| \le \left(\frac{5}{8}\right)^n$$

Since $\lim_{n\to\infty} \left(\frac{5}{8}\right)^n = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem and so $\{F_n\}$ converges uniformly to $F(x)$. Thus

$$\lim_{n\to\infty} \int_1^2 \left(\frac{x^2 + 1}{8}\right)^n \sin nx dx = \int_1^2 \lim_{n\to\infty} F_n(x) dx = \int_1^2 0 dx = 0.$$

□