NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II

Solutions to Tutorial 9

Question 1. Let $f_k(x) = \dfrac{(-1)^k x^k}{1+x^{2k}}$ for $0 < x < \dfrac{2}{3}$. Since

$$|f_k(x)| = \left|\frac{(-1)^k x^k}{1+x^{2k}}\right| \le \frac{\left(\frac{2}{3}\right)^k}{1} = \left(\frac{2}{3}\right)^k$$

for $0 < x < \dfrac{2}{3}$ and the series $\sum_{k=1}^{\infty}\left(\dfrac{2}{3}\right)^k$ converges by the geometric series, the series of functions $\sum_{k=1}^{\infty}\dfrac{(-1)^k x^k}{1+x^{2k}}$ converges uniformly on $\left(0, \dfrac{2}{3}\right)$ by the Weierstrass M-test. Thus the function $F(x) = \sum_{n=1}^{\infty}\dfrac{(-1)^k x^k}{1+x^{2k}}$ is continuous on the interval $\left(0, \dfrac{2}{3}\right)$. □

Question 2 (i). Since

$$|(-1)^k a_k \sin kx| \le |a_k|$$

for $x \in (-\infty, +\infty)$ and $\sum_{k=1}^{\infty}|a_k|$ is convergent, the series of functions $\sum_{k=1}^{\infty} a_k \sin kx$ converges uniformly on $(-\infty, +\infty)$ by the M-test. □

Question 2 (ii). By (i), the series of functions $\sum_{k=1}^{\infty}(-1)^k a_k \sin kx$ converges uniformly on $[0, 2\pi]$. Since each $a_k \sin kx$ is Riemann integrable, we have

$$\int_0^{2\pi}\sum_{k=1}^{\infty}(-1)^k a_k \sin kx dx = \sum_{k=1}^{\infty}\int_0^{2\pi}(-1)^k a_k \sin kx dx = \sum_{k=1}^{\infty} 0 = 0.$$

□

Question 3. Let $f_n(x) = \dfrac{\cos^n x}{n^3}$. Then

(1). Each $f_n'(x) = \dfrac{-\cos^{n-1} x \sin x}{n^2}$ is continuous on $(-\infty, +\infty)$.

(2). The series of functions $\sum_{n=1}^{\infty} f_n(x)$ absolutely converges on $(-\infty, +\infty)$ by the comparison test because $\left|\dfrac{\cos^n x}{n^3}\right| \le \dfrac{1}{n^3}$ and the series $\sum_{n=1}^{\infty}\dfrac{1}{n^3}$ is convergent by the p-series. So it converges pointwise on $(-\infty, +\infty)$.

(3). The series of functions $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $(-\infty, +\infty)$ by the Weierstrass M-test because $\left|\frac{-\cos^{n-1} x \sin x}{n^2}\right| \leq \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series.

Thus the function $f(x) = \sum_{n=1}^{\infty} \frac{\cos^n x}{n^3}$ is differentiable $(-\infty, +\infty)$.

□

Question 4 (i).

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\limsup \sqrt[n]{\left(1+\frac{3}{n}\right)^{n^2}}} = \frac{1}{\lim\limits_{n\to\infty} \left(1+\frac{3}{n}\right)^n} = \frac{1}{e^3}.$$

□

Question 4 (ii).

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{|-3|^{n+1} \cdot n!}{(n+1)! \cdot |-3|^n}} = \frac{1}{\lim\limits_{n\to\infty} \frac{3}{n+1}} = +\infty.$$

□

Question 4 (iii). Since $a_n = \left(\frac{1}{5}\right)^n$ if n is odd and $\left(\frac{1}{6}\right)^n$ if n is even, $\sqrt[n]{|a_n|} = \frac{1}{5}$ if n is odd and $\frac{1}{6}$ if n is even. Thus

$$b_n = \sup_{k\geq n} \sqrt[n]{|a_n|} = \frac{1}{5}$$

and so $\limsup \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} b_n = \frac{1}{5}$. It follows that

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = 5.$$

□

Question 4 (iv). Observe that

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{3^n}{n^2} \cdot (x - \frac{2}{3})^n.$$

Thus

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{3^{n+1} \cdot n^2}{(n+1)^2 \cdot 3^n}} = \frac{1}{\lim\limits_{n\to\infty} \frac{3}{\left(1+\frac{1}{n}\right)^2}} = \frac{1}{3}.$$

□