NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I — Take-home Exam 2 — MA2108 Advanced Calculus II

Tutorial Group:______________________________

Name:____________________ **Matric. No:**____________________

To be submitted during the lecture class on **Tuesday September 27, 2005.** Attach this sheet to your homework as cover page.

There will be a total of 4 homework during the semester.
The full score for each homework is 10 points.
Only your top 3 scores among the 4 homework will be used to count towards your final grade. Late homework will **NOT** be accepted.

Announcement. There will be a **test** on **Thursday October 11**, 2005 during the lecture class. The test will cover materials from Chapter 1 to Chapter 2 of the lecture notes (or roughly, the topics covered in Tutorials 1-7).

The test is a **closed book test**, but you are allowed to bring along **ONE help sheet**.

Definition of a help sheet: A **help sheet** is a piece of paper of size not larger than A4 (21 cm by 30 cm). Anything on the help sheet must be **handwritten** and may be written on both sides of the paper. The handwriting can be as big or as small as the candidate may desire. However, the help sheet must **not** contain any machine printed information of any kind (such as photocopy of a page from either a book or handwritten notes.)

1. Find limit inferior and limit superior of each of the following sequences.

(a) $\left\{\dfrac{n+(-1)^n n^2}{n^2+1}\right\}$.

(b) $\left\{(1+(-1)^n)\sin\dfrac{\pi}{4}\right\}$

(c) $\sqrt[n]{\dfrac{(2n)!}{(n!)^2}}$. [Hint: Recall Proposition 1.8.15 in lecture notes]

2. Let $\{a_n\}$ be the sequence defined recursively by

$$a_1 = 1, \qquad a_{n+1} = 1 + \frac{a_n}{a_n+1} \quad \text{for } n \geq 1.$$

Prove that $\{a_n\}$ is convergent, and find its limit.

3. Let $\{a_n\}$ be a bounded sequence of non-negative real numbers. Show that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \leq 1.$$

[You may use freely the following two facts: (i) If $b_n \leq c_n$ for all $n \geq 1$, then $\overline{\lim_{n\to\infty}}\, b_n \leq \overline{\lim_{n\to\infty}}\, c_n$. (ii) If $\lim_{n\to\infty} b_n$ exists, then $\overline{\lim_{n\to\infty}}\, b_n = \lim_{n\to\infty} b_n$.]

4. If $0 < r < 1$ and $|a_{n+1} - a_n| < r^n$ for all $n \in \mathbb{N}$, prove that $\{a_n\}$ converges.
[Hint: For $m > n$,

$$\begin{aligned} |a_m - a_n| &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \cdots + (a_{n+1} - a_n)| \\ &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \leq r^{m-1} + r^{m-2} + \cdots + r^n. \end{aligned}$$

Show that $\{a_n\}$ is a Cauchy sequence.]