

NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I

MA2108 Advanced Calculus II

Tutorial 1

1. Prove the following limits by using $\epsilon - N$ definition

i) $\lim_{n \rightarrow \infty} \frac{2n^2}{3n^2 + 2} = \frac{2}{3}$.

ii) $\lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = 1$.

2. For each of the following statements, either prove that the statement is true or give a counter example to show that the statement is false:

i) Prove that if $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.
(Hint: Use Theorem 1.4.5)

ii) Give an example of a convergent sequence $\{a_n\}$ and a divergent sequence $\{b_n\}$ such that $\{a_n b_n\}$ converges.

iii) Give an example of two divergent sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n + b_n\}$ converges.

[From Question 3 onwards, you may assume the limits of the standard sequences.]

3. Evaluate the following limits:

i) $\lim_{n \rightarrow \infty} \frac{n^2 + 5n^3 - 1}{3n^3 + 6n + 4}$;

ii) $\lim_{n \rightarrow \infty} \frac{3^n + n^8}{2n^2 + 7^n}$;

iii) $\lim_{n \rightarrow \infty} \sqrt{\frac{n^4 + 4n^3 + 1}{n^3 + 2n^2}}$.

4. Use the Squeeze theorem to find the following limits:

i) $\lim_{n \rightarrow \infty} \frac{1 + |\sin n|}{2n}$;

ii) $\lim_{n \rightarrow \infty} \left(\frac{2n - 5}{3n + 1} \right)^n$.

5. Do the following sequences tend to $+\infty$ or $-\infty$? Justify your answer.

i) $\left\{ \frac{e^n}{n^{100}} \right\}$;

ii) $\left\{ \frac{n}{\ln \frac{1}{n+2}} \right\}$.

6. (a) Let $\{a_k\}$ be a convergent sequence such that

$$a_k \geq 0 \quad \text{for all } k \geq 1.$$

Show that $\lim_{n \rightarrow \infty} a_k \geq 0$.

[Hint: Prove by contradiction.]

- (b) Let $\{b_k\}$ and $\{c_k\}$ be two convergent sequences such that $b_k \leq c_k$ for all $k \geq 1$. Deduce from (i) that

$$\lim_{k \rightarrow \infty} b_k \leq \lim_{k \rightarrow \infty} c_k.$$

- (c) Let $\{b_k\}$ be a convergent sequence. Suppose L is a real number such that $b_k \leq L$ for all $k \geq 1$. Deduce from (b) that

$$\lim_{k \rightarrow \infty} b_k \leq L.$$

- (d) Let $\{a_n\}$ be a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \ln a_n = -\infty$. By using the $\epsilon - N$ definition, show that $\lim_{n \rightarrow \infty} a_n = 0$.