

NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I

MA2108 Advanced Calculus II

Tutorial 2

1. Evaluate the following limits (you may assume the limits of the standard sequences and use the Squeeze theorem, etc.)

- (1) $\lim_{n \rightarrow \infty} \sqrt[3]{\frac{2n^4 + n + 1}{16n^4 + n^2 + 2}}$;
- (2) $\lim_{n \rightarrow \infty} \left(3 + \ln\left(\cos \frac{1}{\sqrt{n}}\right) + \frac{n^2}{1.1^n} \right)$;
- (3) $\lim_{n \rightarrow \infty} \frac{n^4 + 8^n}{9^n + n + 8^n}$;
- (4) $\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{3}}$;
- (5) $\lim_{n \rightarrow \infty} (\sqrt{3n-2} - \sqrt{3n-3})$;
- (6) $\lim_{n \rightarrow \infty} \left(\frac{3 + (-1)^n}{5} \right)^n$;
- (7) $\lim_{n \rightarrow \infty} \frac{7^n + \ln n - n!}{n! + n^2}$;
- (8) $\lim_{n \rightarrow \infty} \frac{n^{100} 100^n}{n!}$;
- (9) $\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}}$;
- (10) $\lim_{n \rightarrow \infty} (3^n + 4^n)^{\frac{1}{n}}$;
- (11) $\lim_{n \rightarrow \infty} n \sin \frac{3}{n}$; (Hint: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.)
- (12) $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n} \right)^{2n}$;
- (13) $\lim_{n \rightarrow \infty} (n^2 + 1)^{\frac{1}{\ln(n+1)}}$

2. Let S and T be two bounded sets of real numbers. Show that $S \cup T$ is also a bounded set.

(Hint: Recall that a bounded set means that this set has an upper bound and a lower bound. The assumption says that S has an upper bound and a lower bound, and T has a (possibly different) upper bound and a (possibly different) lower bound. What you need to do is to find an upper bound and a lower bound for the union $S \cup T$, that is, a **common** upper bound and a **common** lower bound for both S and T . You also have to think how to write down your solution in a *logical* way.)

3. i) Show that if a sequence $\{a_n\}$ is bounded then $\{|a_n|\}$ is bounded.

(Hint: Try to think: Whence you have an upper bound and a lower bound for $\{a_n\}$, how to give an upper bound and a lower bound for $\{|a_n|\}$. Recall also that

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

- ii) Using i) or otherwise, show that if $\lim_{n \rightarrow \infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n \rightarrow \infty} a_n b_n = 0$.

[Warning: $\{b_n\}$ may not be convergent, and thus you cannot use the product rule here.]

(Hint: From (i), $\{b_n\}$ has an upper bound. Recall the $\epsilon - N$ definition.)

- (iii) Give an example of two sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_{n \rightarrow \infty} a_n = 0$, but $\lim_{n \rightarrow \infty} a_n b_n \neq 0$.

Some suggested answers:

1. (1) $1/2\sqrt{2}$ (3) -1 (4) 0 (6) 0