1. Evaluate the following limits (you may assume the limits of the standard sequences and use the Squeeze theorem, etc.)

(1) \[ \lim_{n \to \infty} \sqrt[3]{\frac{2n^4 + n + 1}{16n^4 + n^2 + 2}}; \]

(2) \[ \lim_{n \to \infty} \left( 3 + \ln \left( \frac{1}{\sqrt{n}} \right) + \frac{n^2}{1.1^n} \right); \]

(3) \[ \lim_{n \to \infty} \frac{n^4 + 8^n}{9^n + n + 8^n}; \]

(4) \[ \lim_{n \to \infty} \sqrt[3]{\frac{n}{3}}; \]

(5) \[ \lim_{n \to \infty} (\sqrt{3n} - 2 - \sqrt{3n} - 3); \]

(6) \[ \lim_{n \to \infty} \left( \frac{3 + (-1)^n}{5} \right)^n; \]

(7) \[ \lim_{n \to \infty} \frac{7^n + \ln n - n!}{n^{100}100^n}; \]

(8) \[ \lim_{n \to \infty} \frac{\ln n}{n}; \]

(9) \[ \lim_{n \to \infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}}; \]

(10) \[ \lim_{n \to \infty} (3^n + 4^n) \frac{1}{n}; \]

(11) \[ \lim_{n \to \infty} n \sin \frac{3}{n}; \text{ (Hint: } \lim_{x \to 0} \frac{\sin x}{x} = 1.) \]

(12) \[ \lim_{n \to \infty} \left( 1 - \frac{1}{3n} \right)^{2n}; \]

(13) \[ \lim_{n \to \infty} \frac{(n^2 + 1)^{\frac{1}{n+1}}}{n!}. \]

2. Let \( S \) and \( T \) be two bounded sets of real numbers. Show that \( S \cup T \) is also a bounded set.

(Hint: Recall that a bounded set means that this set has an upper bound and a lower bound. The assumption says that \( S \) has an upper bound and a lower bound, and \( T \) has a (possibly different) upper bound and a (possibly different) lower bound. What you need to do is to find an upper bound and a lower bound for the union \( S \cup T \), that is, a \textbf{common} upper bound and a \textbf{common} lower bound for both \( S \) and \( T \). You also have to think how to write down your solution in a \textit{logical} way.)

3. i) Show that if a sequence \( \{a_n\} \) is bounded then \( \{|a_n|\} \) is bounded.

(Hint: Try to think: Whence you have an upper bound and a lower bound for \( \{a_n\} \), how to give an upper bound and a lower bound for \( \{|a_n|\} \). Recall also that \( |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \)

ii) Using i) or otherwise, show that if \( \lim_{n \to \infty} a_n = 0 \) and \( \{b_n\} \) is bounded, then \( \lim_{n \to \infty} a_nb_n = 0. \)
[Warning: \( \{b_n\} \) may not be convergent, and thus you cannot use the product rule here.]

(Hint: From (i), \( \{|b_n|\} \) has an upper bound. Recall the \( \epsilon - N \) definition.)

(iii) Give an example of two sequences \( \{a_n\} \) and \( \{b_n\} \) such that \( \lim_{n \to \infty} a_n = 0 \), but \( \lim_{n \to \infty} a_n b_n \neq 0 \).

Some suggested answers:

1. (1) \( 1/2\sqrt{2} \)  (3) \(-1\)  (4) \(0\)  (6) \(0\)