

NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I

MA2108 Advanced Calculus II

Tutorial 3

1. Denote the set of rational numbers by  $\mathbb{Q}$ . Consider the set

$$S = \{x \in \mathbb{Q} \mid 0 \leq x < 1\}.$$

Find  $\sup S$  and  $\inf S$ . Justify your answers.

[Suggestion: In justifying that  $\sup S = 1$ , you may like to consider elements in  $S$  which are of the form  $\frac{n}{n+1}$ ,  $n \in \mathbb{N}$ .]

2. Let  $A$  and  $B$  be two non-empty bounded set of real numbers
- Show that  $\sup A \cup B = \max\{\sup A, \sup B\}$ .
  - Is it true that  $\sup A \cap B = \min\{\sup A, \sup B\}$ ? Justify your answer.
3. Let  $A$  and  $B$  be two non-empty bounded set of real numbers.
- Define the set

$$C = \{x \in \mathbb{R} \mid x = a + b \text{ for some } a \in A, b \in B\}.$$

Show that

$$\sup C = \sup A + \sup B.$$

(As an example, if  $A = \{1, 2\}$  and  $B = \{3, 4\}$ , then  $C = \{1 + 3, 1 + 4, 2 + 3, 2 + 4\} = \{4, 5, 6\}$ .)

- Define the set

$$D = \{x \in \mathbb{R} \mid x = a - b \text{ for some } a \in A, b \in B\}.$$

Construct an example to show that in general,

$$\sup D \neq \sup A - \sup B.$$

4. Consider the sequence  $\{a_n\}$  defined recursively by

$$a_1 = 2, \quad a_n = \sqrt{6 + a_{n-1}}, \quad n = 2, 3, 4, \dots$$

- Show that  $2 \leq a_n \leq 3$  for all  $n$ .
  - Show that  $\{a_n\}$  is monotone increasing.
  - Using parts i) and ii), show that  $\{a_n\}$  converges, and find its limit.
5. Consider the sequence  $\{x_n\}$  defined recursively by

$$x_1 = \frac{3}{4}, \quad x_{n+1} = 2x_n - x_n^2, \quad n = 1, 2, 3, \dots$$

Show that  $\{x_n\}$  converges, and find its limit. (Hint: You may like to guess an upper bound for  $\{x_n\}$  by computing the first few terms using your calculator.)