NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II Tutorial 4

1. For each of the following sequences, determine whether the sequence converges or diverges, and find the limit if the sequence converges.
   (i) $\left\{\left|\sin\frac{n\pi}{2}\right|\right\}$.
   (ii) $\left\{|\cos\frac{n\pi}{2}| + |\sin\frac{n\pi}{2}|\right\}$.
   (iii) $\left\{\frac{n+n!}{4^n+n}\right\}$.
2. Let $\{a_n\}$ be a sequence of real numbers. Suppose that both of the two subsequences $\{a_{2k-1}\}$ and $\{a_{2k}\}$ converge and $\lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} a_{2k} = A$. Show that $\lim_{n\to\infty} a_n = A$.
3. Find the $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ of the following sequences.
   (a). $\left\{4+\cos\frac{n\pi}{2}\right\}$.
   (b). $\left\{\frac{1+(-1)^n}{n}\right\}$.
4. (a) Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of real numbers. Prove that
   $$\liminf_{n\to\infty}(a_n+b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$
   [Suggestion: Let $c_n = \inf_{k\geq n}(a_k+b_k)$, $d_n = \inf_{k\geq n} a_k$ and $e_n = \inf_{k\geq n} b_k$. First show that $c_n \geq d_n + e_n$ for all n.
   You may also use freely the following fact: If $\{x_n\}$ and $\{y_n\}$ are two convergent sequences such that $x_n \geq y_n$ for all $n \geq 1$, then $\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n$.]

   (b) Construct an example of two bounded sequences of real numbers $\{a_n\}$ and $\{b_n\}$ such that
   $$\liminf_{n\to\infty}(a_n+b_n) > \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$
5. Let $\{a_n\}$ be a Cauchy sequence of real numbers. Prove that $\{a_n^2 + a_n\}$ is also a Cauchy sequence.
   [Hint: You may use Cauchy's criterion.]