1. Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

(a). \( F_n(x) = \frac{n^2}{n^2 + x^2}, x \in [0, 1] \).

(b). \( F_n(x) = x^n(1 - x), x \in [0, 1] \).

(c). \( f_n(x) = \frac{n \ln x}{x^n}, x \in [1, \infty) \).

(d). \( f_n(x) = \frac{n \ln x \cos nx}{x^n}, x \in [4, \infty) \).

(e). \( F_n(x) = \frac{n^2}{n^2 + x^2}, x \in [0, \infty) \).

2. Let \( \{F_n\} \) be a sequence of functions on an interval \( I \). It is given that \( \{F_n\} \) converges uniformly to some function \( F \) on \( I \). Suppose also that for each \( n \in \mathbb{N} \), there exists a real number \( M_n > 0 \) such that

\[ |F_n(x)| \leq M_n \text{ for all } x \in I. \]

(i) Show that there exists a real number \( M \) such that \( |F(x)| \leq M \) for all \( x \in I \).

[Hint: Recall the definition of uniform convergence.]

(ii) Using part (i) or otherwise, show that \( \{F_nF\} \) also converges uniformly on \( I \).

3. Prove that each of the following series of functions converges uniformly on the indicated interval.

i) \( \sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2}, x \in (-\infty, +\infty) \).

ii) \( \sum_{n=1}^{\infty} \frac{1}{1 + n^2x^2}, x \in [2, \infty) \).

iii) \( \sum_{n=1}^{\infty} \frac{xe^{-nx}}{n^2}, x \in (0, \infty) \).

4. Evaluate the limits. Justify your answers.

i) \( \lim_{n \to \infty} \int_0^1 \frac{n + e^x}{n + x^2} \, dx \).

ii) \( \lim_{n \to \infty} \int_1^2 \left( \frac{x^2 + 1}{8} \right)^n \sin nx \, dx \).