NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

2005/2006 Semester I MA2108 Advanced Calculus II Tutorial 8

1. Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

   (a). $F_n(x) = \dfrac{n^2}{n^2 + x^2}$, $x \in [0, 1]$.

   (b). $F_n(x) = x^n(1 - x)$, $x \in [0, 1]$.

   (c). $f_n(x) = \dfrac{n \ln x}{x^n}$, $x \in [1, \infty)$.

   (d). $f_n(x) = \dfrac{n \ln x \cos nx}{x^n}$, $x \in [4, \infty)$.

   (e). $F_n(x) = \dfrac{n^2}{n^2 + x^2}$, $x \in [0, +\infty)$.

2. Let $\{F_n\}$ be a sequence of functions on an interval $I$. It is given that $\{F_n\}$ converges uniformly to some function $F$ on $I$. Suppose also that for each $n \in \mathbb{N}$, there exists a real number $M_n > 0$ such that

$$|F_n(x)| \leq M_n \quad \text{for all } x \in I.$$

   (i) Show that there exists a real number $M$ such that $|F(x)| \leq M$ for all $x \in I$.
   [Hint: Recall the definition of uniform convergence.]

   (ii) Using part (i) or otherwise, show that $\{F_n F\}$ also converges uniformly on $I$.

3. Prove that each of the following series of functions converges uniformly on the indicated interval.

   i) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2}$, $x \in (-\infty, +\infty)$.

   ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + n^3 x^2}$, $x \in [2, \infty)$.

   iii) $\displaystyle\sum_{n=1}^{\infty} \frac{x e^{-nx}}{n^2}$, $x \in (0, \infty)$.

4. Evaluate the limits. Justify your answers.

   i) $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{n + e^x}{n + x^2} dx$.

   ii) $\displaystyle\lim_{n\to\infty} \int_1^2 \left(\frac{x^2 + 1}{8}\right)^n \sin nx dx$.