Foulkes Conjecture in Representation Theory and its relations in Rational Homotopy Theory (Jie Wu)

Let Σ_n be the symmetric group and let F be a field. Let $(\xi^m)^k = F\otimes_{\Sigma_m\wr\Sigma_k} F(\Sigma_{km})$, where $\Sigma_m\wr\Sigma_k$ is the wreath product. Let $\{\alpha^\lambda\}$ be the set of irreducible Σ_{km}-module.

Foulkes Conjecture: Let $k<m$. Suppose that F is of characteristic 0. Then

$$\dim(\mathrm{Hom}_{\Sigma_{\mathrm{km}}}((\xi^{\mathrm{k}})^{\mathrm{m}},\alpha^\lambda))\leq\dim(\mathrm{Hom}_{\Sigma_{\mathrm{km}}}((\xi^{\mathrm{m}})^{\mathrm{k}},\alpha^\lambda))$$

for any λ.

Note: The following statements are equivalent:

1) Foulkes Conjecture.
2) Let $k<m$ and let F be of characteristic zero. Then $(\xi^k)^m$ is isomorphic to a submodule of $(\xi^m)^k$.
3) Let $k<m$ and let F be of characteristic zero. Then $(\xi^k)^m$ is an Σ_{km}-summand of $(\xi^m)^k$.
4) Let $k<m$ and let F be of characteristic zero. Then $(\xi^k)^m$ is isomorphic to a quotient Σ_{km}-module of $(\xi^m)^k$.
5) Let $k<m$ and let F be of characteristic zero. Then $\Lambda_m(\Lambda_k(V))$ is a **functorial** summand of $\Lambda_k(\Lambda_m(V))$ for any vector space V, where $\Lambda_n(V)$ is the set of free homogeneous polynomial of degree n generated by V.
6) Let $k<m$ and let F be of characteristic zero. Then $\Lambda_m(\Lambda_k(V))$ is a **functorial** quotient of $\Lambda_k(\Lambda_m(V))$ for any vector space V.
7) Let $k<m$ and let F be of characteristic zero. Then there is a **functorial** monomorphism from $\Lambda_m(\Lambda_k(V))$ to $\Lambda_k(\Lambda_m(V))$ for any vector space V.
8) Let $k<m$ and let X runs over all suspension spaces. Then $\bar{\mathrm{Sp}}^m(\bar{\mathrm{Sp}}^k(X))$ is a **functorial rational** retract of $\bar{\mathrm{Sp}}^k(\bar{\mathrm{Sp}}^m(X))$ for any suspension X, where $\bar{\mathrm{Sp}}^n(X)$ is the **reduced symmetric product** of X.

By rational representation theory of symmetric groups, statements 1-4 are equivalent each other.

Statements 5-7 follow from statements 1-4 because

$$\Lambda_m(\Lambda_k(V))=(\xi^k)^m\otimes_{\Sigma_{km}}V^{\otimes km}$$

for any k,m,V, where Σ_n acts on $V^{\otimes n}$ by permuting factors.

Statements 1-4 follow from statements 5-7 by choosing V to be the km-dimensional vector space with a basis $x_1,\cdots,x_{km}$. Notice that $(\xi^k)^m$ is isomorphic to the submodule $\Lambda_m(\Lambda_k(V))$ generated by the monomials $x_{\sigma_1}x_{\sigma_2}\cdots x_{\sigma_{km}}$ for $\sigma\in\Sigma_{km}$.

Statement 8 follows from statements 1-7 by using geometric realization. Statements 1-7 follow from statement 8 by taking homology.