Simplicial Objects

Jie Wu

Department of Mathematics
National University of Singapore

December 20-23, 2007, Fudan University
Simplicial Objects and Homotopy Groups

Δ-Objects and Homology

Simplicial Sets and Homotopy

Simplicial Groups

Braids and Homotopy Groups
Definitions

- A **Δ-set** means a sequence of sets $X = \{X_n\}_{n \geq 0}$ with faces $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, such that

$$d_i d_j = d_j d_{i+1} \quad (1)$$

for $i \geq j$, which is called the **Δ-identity**.

- One can use coordinate projections for catching Δ-identity:

$$d_i : (x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

- A **Δ-map** $f : X \to Y$ means a sequence of functions $f : X_n \to Y_n$

for each $n \geq 0$ such that $f \circ d_i = d_i \circ f$.

- A Δ-set $G = \{G_n\}_{n \geq 0}$ is called a **Δ-group** if each $G_n$ is a group, and each face $d_i$ is a group homomorphism.
**Δ-set is a contravariant functor**

- Let $\mathcal{O}^+$ be the category whose objects are finite ordered sets and whose morphisms are functions $f : X \to Y$ such that $f(x) < f(y)$ if $x < y$.
- The objects in $\mathcal{O}^+$ are given by $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and the morphisms in $\mathcal{O}^+$ are generated by $d^i : [n - 1] \to [n]$ the ordered embedding missing $i$.
- **Proposition.** Let $S$ denote the category of sets. $\Delta$-sets are one-to-one correspondent to contravariant functors from $\mathcal{O}^+$ to $S$.
- a $\Delta$-group means a contravariant functor from $\mathcal{O}^+$ to the category of groups.
- More abstractly, for any category $\mathcal{C}$, a $\Delta$-object over $\mathcal{C}$ means a contravariant functor from $\mathcal{O}^+$ to $\mathcal{C}$. In other words, a $\Delta$-object over $\mathcal{C}$ means a sequence of objects over $\mathcal{C}$, $X = \{X_n\}_{n \geq 0}$ with faces $d_i : X_n \to X_{n-1}$ as morphisms in $\mathcal{C}$.
Example of $\Delta$-sets

- The $n$-simplex $\Delta^+[n]$, as a $\Delta$-set, is as follows:

  $$\Delta^+[n]_k = \{(i_0, i_1, \ldots, i_k) \mid 0 \leq i_0 < i_1 < \cdots < i_k \leq n\}$$

  for $k \leq n$ and $\Delta^+[n]_k = \emptyset$ for $k > n$. The face $d_j: \Delta^+[n]_k \to \Delta^+[n]_{k-1}$ is given by

  $$d_j(i_0, i_1, \ldots, i_k) = (i_0, i_1, \ldots, \hat{i}_j, \ldots, i_k),$$

  that is deleting $i_j$. Let $\sigma_n = (0, 1, \ldots, n)$. Then any elements in $\Delta[n]$ can be written an iterated face of $\sigma_n$.

- **Proposition.** Let $X$ be a $\Delta$-set and let $x \in X_n$ be an element. Then there exists a unique $\Delta$-map, called the **representing map** of $x$,

  $$f_x: \Delta^+[n] \to X$$

  such that $f_x(\sigma_n) = x$. 
Abstract Simplicial Complexes

• An **abstract simplicial complex** \( K \) is a collection of finite nonempty sets, such that if \( A \) is an element in \( K \), so is every nonempty subset of \( A \).

• The element \( A \) of \( K \) is called a **simplex** of \( K \); its **dimension** is one less than the number of its elements. Each nonempty subset of \( A \) is called a **face** of \( A \).

• Let \( K \) be an abstract simplicial complex with vertices ordered. Let \( K_n \) be the set of \( n \)-simplices of \( K \). Define the faces \( d_i: K_n \rightarrow K_{n-1}, \quad 0 \leq i \leq n \), as follows. If \( \{a^0, a^1, \ldots, a^n\} \) is an \( n \)-simplex of \( K \) with \( a^0 < a^1 < \cdots < a^n \), then define

\[
d_i\{a^0, a^1, \ldots, a^n\} = \{a^0, a^1, \ldots, a^{i-1}, a^{i+1}, \ldots, a^n\}.
\]

• **Proposition.** Let \( K^\Delta = \{K_n\}_{n \geq 0} \) with faces defined as above. Then \( K^\Delta \) is a \( \Delta \)-set.
Polyhedral

- A $\Delta$-set $X$ is called **polyhedral** if there exists an abstract simplicial complex $\mathcal{K}$ such that $X \cong \mathcal{K}^\Delta$.

- **Exercise** In general, a $\Delta$-set may not be polyhedral. Let $X = \Delta^+[1] \cup_{\Delta^+[1]} \Delta^+[1]$ be the union of two copies of $\Delta^+[1]$ by identifying the vertices. Show that $X$ is not polyhedral.

- Let $X$ be a $\Delta$-set and let $2^{X_0}$ be the set of all subsets of $X_0$. Define

$$
\phi: \bigsqcup_{n \geq 0} X_n \longrightarrow 2^{X_0}
$$

by setting $\phi(x) = \{f_x(0), f_x(1), \ldots, f_x(n)\}$ for $x \in X_n$.

- **Theorem.** Let $X$ be a $\Delta$-set. Then $X$ is polyhedral if and only if the following holds:

1. There exists an order of $X_0$ such that, for each $x \in X_n$,

$$
f_x(0) \leq f_x(1) \leq \cdots \leq f_x(n).
$$

2. The function $\phi: \bigcup_{n \geq 0} X_n \longrightarrow 2^{X_0}$ is one-to-one.
\(\Delta\)-complex

- The standard **geometric** \(n\)-simplex \(\Delta^n\) is defined by
\[
\Delta^n = \{ (t_0, t_1, \ldots, t_n) \mid t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1 \}.
\]

Define \(d^i : \Delta^{n-1} \to \Delta^n\) by setting
\[
d^i(t_0, t_1, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_{i}, \ldots, t_{n-1}).
\]

- A **\(\Delta\)-complex structure** on a space \(X\) is a collection of maps
\[
\mathcal{C}(X) = \{ \sigma_\alpha : \Delta^n \to X \mid \alpha \in J_n \ n \geq 0 \}.
\]
such that

1. \(\sigma_\alpha|_{\text{Int}(\Delta^n)} : \text{Int}(\Delta^n) \to X\) is injective, and each point of \(X\) is in the image of exactly one such restriction \(\sigma_\alpha|_{\text{Int}(\Delta^n)}\).

2. For each \(\sigma_\alpha \in \mathcal{C}(X)\), each face
\[
\sigma_\alpha \circ d^i \in \mathcal{C}(X).
\]

3. A set \(A \subseteq X\) is open if and only if \(\sigma_\alpha^{-1}(A)\) is open in \(\Delta^n\) for each \(\sigma_\alpha \in \mathcal{C}(X)\).
Δ-set from Δ-complex

- Let $X$ be a $Δ$-complex. Define

  $$C_n^Δ(X) = \{\sigma_\alpha : \Delta^n \to X \mid \alpha \in J_n\} \subseteq C(X)$$

  with $d_i : C_n^Δ(X) \to C_{n-1}^Δ(X)$ given by

  $$d_i(\sigma_\alpha) = \sigma_\alpha \circ d^i$$

  for $0 \leq i \leq n$.

- $C^Δ(\Delta) = \{C_n^Δ(\Delta)\}_{n \geq 0}$ is a $Δ$-set.
Geometric Realization of $\Delta$-sets

- Let $K$ be a $\Delta$-set. The **geometric realization** $|K|$ of $K$ is defined to be

$$|K| = \bigsqcup_{x \in K_n, n \geq 0} (\Delta^n, x) / \sim = \bigsqcup_{n=0}^\infty \Delta^n \times K_n / \sim,$$

where $(\Delta^n, x)$ is $\Delta^n$ labeled by $x \in K_n$ and $\sim$ is generated by $(z, d_i x) \sim (d^i z, x)$ for any $x \in K_n$ and $z \in \Delta^{n-1}$ labeled by $d_i x$.

- For any $x \in K_n$, let $\sigma_x : \Delta^n = (\Delta^n, x) \to |K|$ be the canonical characteristic map. The topology on $K$ is defined by $A \subseteq |K|$ is open if and only if the pre-image $\sigma_x^{-1}(A)$ is open in $\Delta^n$ for any $x \in K_n$ and $n \geq 0$.

- **Proposition.** Let $K$ be a $\Delta$-set. Then $|K|$ is $\Delta$-complex.
Homology of $\Delta$-sets

- A chain complex of groups means a sequence $C = \{C_n\}$ of groups with differential $\partial_n: C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1}$ is trivial, that is $\text{Im}(\partial_{n+1}) \subseteq \text{Ker}(\partial_n)$ and so the homology is defined by
  $$H_n(C) = \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1}),$$
  which is a coset in general.

- **Proposition.** Let $G$ be a $\Delta$-abelian group. Define
  $$\partial_n = \sum_{i=0}^{n} (-1)^i d_i: G_n \to G_{n-1}.$$
  Then $\partial_{n-1} \circ \partial_n = 0$, that is, $G$ is a chain complex under $\partial_*$.

- Let $X$ be a $\Delta$-set. The homology $H_*(X; G)$ of $X$ with coefficients in an abelian group $G$ is defined by
  $$H_*(X; G) = H_*(\mathbb{Z}(X) \otimes G, \partial_*),$$
  where $\mathbb{Z}(X) = \{\mathbb{Z}(X_n)\}_{n \geq 0}$ and $\mathbb{Z}(X_n)$ is the free abelian group generated by $X_n$. 
Simplicial and Singular Homology

• Let $X$ be a $\Delta$-complex. Then **simplicial homology** of $X$ with coefficients in an abelian group $G$ is defined by

$$H^\Delta_*(X; G) = H_*(C^\Delta_*(X); G).$$

• For any space $X$, define

$$S_n(X) = \text{Map}(\Delta^n, X)$$

be the set of all continuous maps from $\Delta^n$ to $X$ with

$$d_i = d^{i*} : S_n(X) = \text{Map}(\Delta^n, X) \longrightarrow S_{n-1}(X) = \text{Map}(\Delta^{n-1}, X).$$

for $0 \leq i \leq n$. Then $S_*(X) = \{S_n(X)\}_{n \geq 0}$ is a $\Delta$-set. The **singular homology** $H_*(X; G) = H_*(S_*(X); G).$
Definition

- **A simplicial set** means a $\Delta$-set $X$ together with a collection of **degeneracies** $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$, such that

  $$d_j d_i = d_{i-1} d_j$$  \hspace{1cm} (2)

  for $j < i$,

  $$s_j s_i = s_{i+1} s_j$$  \hspace{1cm} (3)

  for $j \leq i$ and

  $$d_j s_i = \begin{cases} 
  s_{i-1} d_j & j < i \\
  \text{id} & j = i, i+1 \\
  s_i d_{j-1} & j > i+1.
  \end{cases}$$ \hspace{1cm} (4)

  The three identities for $d_i d_j$, $s_j s_i$ and $d_i s_j$ are called the **simplicial identities**.

- **A simplicial map** $f$ means a sequence of functions $f : X_n \to Y_n$ such that $d_i f = f d_i$ and $s_i f = f s_i$.

- One can use **deleting-doubling coordinates** for catching simplicial identities.
Simplicial Set is a Functor

• Let $\mathcal{O}$ be the category: objects: finite ordered sets; morphisms: functions $f : X \to Y$ such that $f(x) \leq f(y)$ if $x < y$. The objects in $\mathcal{O}$ are $[n] = \{0, \ldots, n\}$ for $n \geq 0$, same as the objects in $\mathcal{O}^+$. The morphisms in $\mathcal{O}$ are generated by $d^i$, which is defined in $\mathcal{O}^+$, and the following morphism $s^i : [n + 1] \to [n]

\[
s^i = \begin{pmatrix}
0 & 1 & \cdots & i - 1 & i & i + 1 & i + 2 & \cdots & n + 1 \\
0 & 1 & \cdots & i - 1 & i & i & i + 1 & \cdots & n
\end{pmatrix}
\]

for $0 \leq i \leq n$, that is, $s^i$ hits $i$ twice.

• Exercise. simplicial sets are one-to-one correspondent to contravariant functors from $\mathcal{O}$ to $\mathcal{S}$.

• Let $\mathcal{C}$ be a category. A simplicial object over $\mathcal{C}$ means a contravariant functor from $\mathcal{O}$ to $\mathcal{C}$, i.e. a sequence of objects $X = \{X_n\}_{n \geq 0}$ with face morphisms $d_i : X_n \to X_{n-1}$ and degeneracy morphisms $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$, such that the three simplicial identities hold.
Geometric Realization

- Let $X$ be a simplicial set. Then its geometric realization $|X|$ is a CW-complex defined by

$$|X| = \coprod_{n \geq 0} (\Delta^n, x)/ \sim = \coprod_{n=0}^{\infty} \Delta^n \times X_n / \sim,$$

where $(\Delta^n, x)$ is $\Delta^n$ labeled by $x \in X_n$ and $\sim$ is generated by

$$(z, d_ix) \sim (d^i z, x)$$

for any $x \in X_n$ and $z \in \Delta^{n-1}$ labeled by $d_ix$, and

$$(z, s_ix) \sim (s^i z, x)$$

for any $x \in X_n$ and $z \in \Delta^{n+1}$ labeled by $s_ix$. Note that the points in $(\Delta^{n+1}, s_ix)$ and $(\Delta^{n-1}, d_ix)$ are identified with the points in $(\Delta^n, x)$.

- $|X|$ is a CW-complex.
The $n$-simplex $\Delta[n]$, as a simplicial set, is as follows:

$$\Delta[n]_k = \{(i_0, i_1, \ldots, i_k) \mid 0 \leq i_0 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

for $k \leq n$. The face $d_j: \Delta[n]_k \to \Delta[n]_{k-1}$ is given by

$$d_j(i_0, i_1, \ldots, i_k) = (i_0, i_1, \ldots, \hat{i}_j, \ldots, i_k),$$

that is deleting $i_j$. The degeneracy $s_j: \Delta[n]_k \to \Delta[n]_{k+1}$ is defined by

$$s_j(i_0, i_1, \ldots, i_k) = (i_0, i_1, \ldots, i_j, i_j, \ldots, i_k),$$

that is doubling $i_j$. Let $\sigma_n = (0, 1, \ldots, n) \in \Delta[n]_n$. Then any elements in $\Delta[n]$ can be written as iterated compositions of faces and degeneracies of $\sigma_n$.

Let $X$ be a simplicial set and let $x \in X_n$ be an element. Then there exists a unique simplicial map, called representing map of $x$,

$$f_x: \Delta[n] \to X$$
Basic Constructions

- **Cartesian Product:** $(X \times Y)_n = X_n \times Y_n$.
  $|X \times Y| \cong |X| \times |Y|$ under compact generated topology.

- **Milnor Theorem.** The geometric realization of a simplicial group is a topological group (under compactly generated topology).

- **Wedge:** $(X \vee Y)_n = X_n \vee Y_n$, union at the basepoint.
  $|X \vee Y| \cong |X| \vee |Y|$.

- **Smash Product:** $X \wedge Y = (X \times Y)/(X \vee Y)$.
  $|X \wedge Y| \cong |X| \wedge |Y|$ under compactly generated topology.
Mapping Space

- Let $\sigma_n = (0, 1, \ldots, n) \in \Delta[n]$ be the nondegenerate element.
  
  $d^i = f_{d_i\sigma_n} : \Delta[n-1] \to \Delta[n]$ 
  
  $s^i = f_{s_i\sigma_n} : \Delta[n+1] \to \Delta[n]$ 

- Let $\text{Map}(X, Y)_n = \text{Hom}_S(X \times \Delta[n], Y)$ and let
  
  $d_i = (\text{id}_X \times d^i)^* : \text{Map}(X, Y)_n \to \text{Hom}_S(X \times \Delta[n], Y)$ 
  
  $s_i = (\text{id}_X \times s^i)^* : \text{Map}(X, Y)_n \to \text{Hom}_S(X \times \Delta[n+1], Y)$

for $0 \leq i \leq n$. $\text{Map}(X, Y) = \{ \text{Map}(X, Y)_n \}_{n \geq 0}$ with $d_i$ and $s_i$ is a simplicial set, which is called the mapping space from $X$ to $Y$. 
Homotopy

- Let $I = \Delta[1]$. As a simplicial set,

$$I_n = \{ (\overbrace{t_{\frac{i}{n+1}}}, \ldots, \overbrace{t_{\frac{i}{n+1}}}) \mid 0 \leq i \leq n+1 \}$$

Given a simplicial set $X$, the simplicial subsets $X \times 0$ and $X \times 1$ of $X \times I$ are given by $(X \times 0)_n = \{ (x, t_0) \mid x \in X_n \}$, $(X \times 1)_n = \{ (x, t_1) \mid x \in X_n \}$.

- Let $f, g : X \rightarrow Y$ be simplicial maps. We call $f$ homotopic to $g$ if there is a simplicial map

$$F : X \times I \rightarrow Y$$

such that $F|_{X \times 0} = f$ and $F|_{X \times 1} = g$ in which case write $f \simeq g$. If $A$ is a simplicial subset of $X$ and $f, g : X \rightarrow Y$ are simplicial maps such that $f|_A = g|_A$, we call $f$ homotopic to $g$ relative to $A$, denoted by $f \simeq g \text{ rel } A$, if there is a homotopy $F : X \times I \rightarrow Y$ such that $F|_{X \times 0} = f$, $F|_{X \times 1} = g$ and $F|_{A \times I} = f$. 
Fibrant Simplicial Set

- Let $X$ be a simplicial set. The elements

$$x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in X_{n-1}$$

are called **matching faces** with respect to $i$ if

$$d_j x_k = d_k x_{j+1}$$

for $j \geq k$ and $k, j + 1 \neq i$. A simplicial set $X$ is called **fibrant** (or **Kan complex**) if it satisfies the following homotopy extension condition for each $i$:

Let $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in X_{n-1}$ be any elements that are matching faces with respect to $i$. Then there exists an element $w \in X_n$ such that $d_j w = x_j$ for $j \neq i$.

- **Theorem.** Let $X$ and $Y$ be simplicial sets. If $Y$ is fibrant, then $[X, Y] = \text{Hom}_S(X, Y)/\sim$ is the same as $[|X|, |Y|]$. 
Simplicial Fibration

- A simplicial map \( p: E \to B \) is called a (Kan) fibration if for every commutative diagram of simplicial maps

\[
\begin{array}{ccc}
\Lambda^k[n] & \xrightarrow{i} & E \\
\downarrow & & \downarrow^{p} \\
\Delta[n] & \xrightarrow{\theta} & B
\end{array}
\]

there is a simplicial map \( \theta: \Delta[n] \to X \) (the dotted arrow) making the diagram commute, where \( i \) is the inclusion of \( \Lambda^k[n] \) in \( \Delta[n] \). Let \( v \in B_0 \) be a vertex. Then \( F = p^{-1}(v) \), that is \( F = \{ p^{-1}(s^n v) \}_{n \geq 0} \) is called a fibre of \( p \) over \( v \).

- Let \( p: E \to B \) be a fibration and let \( F = p^{-1}(\ast) \) be the fibre. Suppose that \( E \) or \( B \) is fibrant. Then there is a long exact sequence on homotopy groups.
Moore Paths and Moore Loops

• Let $X$ be a pointed simplicial set. The **Moore Path** is defined by setting

$$(PX)_n = \{ x \in X_{n+1} \mid d_1 d_2 \cdots d_{n+1} x = \ast \}$$

with faces $d^P_i = d^{X}_{i+1} |_{(PX)_n} : (PX)_n \to (PX)_{n-1}$ and
degeneracies $s^P_i = s^{X}_{i+1} |_{(PX)_n} : (PX)_n \to (PX)_{n+1}$ for $0 \leq i \leq n$.

• **Proposition.** $|PX|$ is contractible. If $X$ is fibrant, $PX \to X$ is a fibration.

• Let $X$ be a pointed fibrant simplicial set. The **Moore loop** $\Omega X$ is defined to be the fibre of $p_X : PX \to X$. More precisely,

$$(\Omega X)_n = \{ x \in X_{n+1} \mid d_1 d_2 \cdots d_{n+1} x = \ast \text{ and } d_0 x = \ast \}$$

with faces $d^\Omega_i = d^{X}_{i+1}$ and degeneracies $s^\Omega_i = s^{X}_{i+1}$ for $0 \leq i \leq n$. 
Moore Postnikov System

Let $X$ be a simplicial set. For each $n \geq 0$, define the equivalence relation $\sim_n$ on $X$ as follows: For $x, y \in X_q$, we call $x \sim_n y$ if each iterated face of $x$ of dimension $\leq n$ agrees with the correspondent iterated face of $y$. Define

$$P_nX = X / \sim_n$$

for each $n \geq 0$. Then we have the tower

$$X \longrightarrow \cdots \longrightarrow P_nX \longrightarrow P_{n-1}X \longrightarrow \cdots \longrightarrow P_0X$$

called Moore-Postnikov system of $X$ or coskeleton filtration of $X$. 
Theorem

Let $X$ be a pointed fibrant simplicial set and let

$$p_n: X \to P_n(X)$$

be the projection. Then

- each $p_n: X \to P_nX$ is a fibration, each $P_nX$ is fibrant, and for each $n \geq m$, the projection $P_nX \to P_mX$ is a fibration.
- $p_n^*: \pi_q(X) \to \pi_q(P_nX)$ is an isomorphism for $q \leq n$.
- Let $q > n$ and let $x$ be a spherical element in $(P_nX)_q$. Then $x = \ast$. In particular, $\pi_q(P_nX) = 0$ for $q > n$.
- Let $E_nX$ be the fibre of the projection $P_nX \to P_{n-1}X$. Then $E_nX$ is an Eilenberg-MacLane complex of $K(\pi_n(X), n)$. 
Moore Chain Complex

- Let $G$ be a simplicial group. Define

$$N_n G = \bigcap_{j=1}^{n} \ker(d_j : G_n \to G_{n-1}).$$

Then $NG = \{N_n G, d_0\}$ is a chain complex.

- **Moore Theorem.** $H_n(NG) = \pi_n(G) \cong \pi_n(|G|)$.
- **Moore cycles:** $Z_n G = \bigcap_{j=0}^{n} \ker(d_j : G_n \to G_{n-1})$.
- **Moore boundaries:** $B_n G = d_0(N_{n+1} G)$.
- $\pi_n(|G|) = Z_n G / B_n G$.
- The key point here is that the geometric homotopy group $\pi_n(|G|)$ can be given by the homology of the chain complex $NG$. 

Abelian Simplicial Groups

- Let $G$ be a simplicial abelian groups. Define

$$\partial_n : G_n \to G_{n-1}$$

by setting

$$\partial_n(x) = \sum_{j=0}^{n} (-1)^j d_j x.$$  

As we have seen for abelian $\Delta$-groups, $(G, \partial)$ becomes a chain complex. Thus for a simplicial abelian group $G$ we have two chain complexes $(NG, d_0)$ and $(G, \partial)$. Let $x \in N_n G$. Then $\partial(x) = d_0 x$ because $d_j x = 0$ for $j > 0$, and so $(NG, d_0)$ is a chain subcomplex of $(G, \partial)$.

- **Theorem.** Let $G$ be a simplicial abelian group. Then the inclusion

$$(NG, d_0) \to (G, \partial)$$

induces an isomorphism on homology

$$H_*(NG, d_0) \cong H_*(G, \partial).$$
Hurewicz Theorem

- Let $X$ be a simplicial set. Let $\mathbb{Z}(X) = \{\mathbb{Z}(X_n)\}_{n \geq 0}$ be the sequence of the free abelian group generated by $X_n$. Then $\mathbb{Z}(X)$ is a simplicial abelian group. The integral homology of $X$ is defined by

$$H_*(X) = \pi_*(\mathbb{Z}(X)) \cong H_*(\mathbb{Z}(X), \partial).$$

- **Hurewicz homomorphism** The map $\tilde{j}: X \to \mathbb{Z}(X), \ x \mapsto x - *$ induces a group homomorphism $h_n = \tilde{j}_*: \pi_n(X) \to \pi_n(\mathbb{Z}(X)) = H_n(X)$ for any fibrant simplicial set $X$ and $n \geq 1$.

- Let $X$ be a fibrant simplicial set with $\pi_i(X) = 0$ for $i < n$ with $n \geq 2$. Then $\tilde{H}_i(X) = 0$ for $i < n$ and

$$h_n: \pi_n(X) \to \tilde{H}_n(X)$$

is an isomorphism.
Group Homology

- Let $G$ be a monoid. Let $WG$ be the simplicial set given by

$$(WG)_n = \{(g_0, g_1, \ldots, g_n) \mid g_i \in G\} = G^{n+1}$$

with faces and degeneracies given by

$$d_i(g_0, \ldots, g_n) = \begin{cases} (\ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots) & \text{if } i < n \\ (g_0, \ldots, g_{n-1}) & \text{if } i = n \end{cases}$$

$$s_i(g_0, \ldots, g_n) = (g_0, g_1, \ldots, g_i, e, g_{i+1} \ldots, g_n).$$

- Let $R$ be a commutative ring and let $M$ be an $R(G)$-module. Then we have the simplicial abelian group

$$R(WG) \otimes_{R(G)} M.$$ 

- The homology of $G$ with coefficients in $M$ is then defined by

$$H_*(G; M) = \pi_*(R(WG) \otimes_{R(G)} M) \cong H_*(R(WG) \otimes_{R(G)} M, \partial).$$
Milnor’s Construction

- A pointed set $S$ means a set $S$ with a basepoint $\ast$. Denote by $F[S]$ the free group generated by $S$ subject to the relation $\ast = 1$.

- Let $X$ be a pointed simplicial set. Milnor’s construction is the simplicial group $F[X]$, where $F[X]_n = F[X_n]$ with face and degeneracy homomorphisms induced by the faces and degeneracies of $X$.

- **Theorem.** $|F[X]| \simeq \Omega \Sigma |X|$. The loop space of the suspension of $|X|$.

- **James’ Construction.** $J(X)$ is the free monoid generated by $X$ subject to the relation $\ast = 1$.

- **Theorem.** If $|X|$ is path-connected, then the inclusion $|J(X)| \to |F[X]|$ is a homotopy equivalence.
Kan’s Construction

- For a reduced simplicial set $X$, let $GX$ be the simplicial group defined by
  1. $(GX)_n$ is the quotient group of $X_{n+1}$ subject to the relations:
     \[ s_0x = 1 \]
     for every $x \in X_n$. Thus, as a group, $(GX)_n$ is the free group generated by $X_{n+1} \setminus s_0(X_n)$; or $(GX)_n = F[X_{n+1}/s_0(X_n)]$.
  2. The face and degeneracy operators are the group homomorphisms such that
     \[ d_0^{GX}x = (d_1x)(d_0x)^{-1}, \]
     \[ d_i^{GX}x = d_{i+1}x \text{ for } i > 0, \]
     \[ s_i^{GX}x = s_{i+1}x \]
     for $x \in X_{n+1}$.
- **Theorem.** $|GX| \simeq \Omega|X|$. 

Central Extension Theorem (-)

- **Proposition.** The action of $G_0$ on $G$ by conjugation induces an action of $\pi_0(G)$ on $\pi_n(G)$ for each $n \geq 0$.

- We call a simplicial group $G$ **$n$-simple** if $\pi_0(G)$ acts trivially on $\pi_n(G)$. A simplicial group $G$ is called **simple** if it is $n$-simple for every $n$.

- **Theorem.** Let $G$ be a simplicial group and let $n \geq 0$. Suppose that $G$ is $n$-simple. Then the homotopy group $\pi_n(G)$ is contained in the center of $G_n/B_nG$.

- **Theorem.** Let $G$ be a reduced simplicial group such that $G_n$ is a free group for each $n$. Then there exits a unique integer $\gamma_G > 0$ such that $G_n = \{1\}$ for $n < \gamma_G$ and $\text{rank}(G_n) \geq 2$ for $n > \gamma_G$. Furthermore,

  $$\pi_n(G) \cong \mathbb{Z}(G_n/B_nG)$$

  for $n \neq \gamma_G + 1$. 
Abelianization of Moore Chains

- Let $G$ be a group. Write $G^{ab} = G/[G, G]$ for the **abelianization** of the group $G$. Now given a simplicial group $G$, we have a chain complex of (non-abelian in general) groups $(Ng, d_0)$. Take the abelianization $(N_nG)^{ab}$ of $N_nG$ for each $n$, the differential $d_0 : N_nG \rightarrow N_{n-1}G$ induces a homomorphism $d_0^{ab} : (N_nG)^{ab} \rightarrow (N_{n-1}G)^{ab}$ and so a chain complex of abelian groups $((Ng)^{ab}, d_0^{ab})$.

- **New Theorem.** Let $G$ be a simplicial group such that (1) each $G_n$ is a free group and (2) $\pi_0(G)$ acts trivially on each $\pi_n(G)$. Then there is a decomposition

$$H_n(((Ng)^{ab}) \cong \pi_n(G) \oplus A_n,$$

where $A_n$ is a free abelian group. In particular, $\text{Tor}(H_\ast(((Ng)^{ab})) \cong \text{Tor}(\pi_\ast(G))$. 

Simplicial Group $F[S^1] \simeq \Omega S^2$

Let $S^1$ be the simplicial 1-sphere. The elements in $S_n^1$ can be listed as follows.

$S_0^1 = \{\ast\}$, $S_1^1 = \{s_0\ast, \sigma\}$, $S_2^1 = \{s_0^2\ast, s_0\sigma, s_1\sigma\}$,

$S_3^1 = \{s_0^3\ast, s_2s_1\sigma, s_2s_0\sigma, s_1s_0\sigma\}$, and in general

$S_{n+1}^1 = \{s_0^{n+1}\ast, x_0, \ldots, x_n\}$, where $x_j = s_n \cdots \hat{s}_j \cdots s_0\sigma$. The face

$$d_i: S_{n+1}^1 = \{\ast, x_0, \ldots, x_n\} \longrightarrow S_n^1 = \{\ast, x_0, \ldots, x_{n-1}\}$$

is given by $d_is_0^{n+1}\ast = s_0\ast$ and

$$d_ix_j = \begin{cases} 
  s_0^{n}\ast & \text{if } j = i = 0 \text{ or } i = j + 1 = n + 1 \\
  x_j & \text{if } j < i \\
  x_{j-1} & \text{if } j \geq i.
\end{cases}$$

Similarly,

$$s_is_j = s_is_n \cdots \hat{s}_j \cdots s_0\sigma = \begin{cases} 
  x_j & \text{if } j < i \\
  x_{j+1} & \text{if } j \geq i.
\end{cases}$$
Simplicial Group $F[S^1] \cong \Omega S^2$

- $F[S^1]$ is a simplicial group model for $\Omega S^2$. As a sequence of groups, the group $F[S^1]_n$ is the free group of rank $n$ generated by $x_0, x_1, \ldots, x_{n-1}$ with faces as above.

- Let $y_0 = x_0 x_1^{-1}, \ldots, y_{n-1} = x_{n-1} x_n^{-1}$ and $y_n = x_n$ in $F[S^1]_{n+1}$. Clearly \{ $y_0, y_1, \ldots, y_n$ \} is a set of free generators for $F[S^1]_{n+1}$ with $d_i y_j$ ($0 \leq i \leq n+1, -1 \leq j \leq n$) given by

$$d_i y_j = d_i(x_j x_{j+1}^{-1}) = \begin{cases} y_j & \text{if } j < i - 1 \\ 1 & \text{if } j = i - 1 \\ y_{j-1} & \text{if } j \geq i, \end{cases}$$

$$s_i y_j = \begin{cases} y_j & \text{if } j < i - 1 \\ y_j y_{j+1} & \text{if } j = i - 1 \\ y_{j+1} & \text{if } j \geq i, \end{cases}$$

where $y_{-1} = (y_0 y_1 \cdots y_{n-1})^{-1}$ and in this formula $x_{n+1} = 1$. 
Simplicial Group $F[S^1] \simeq \Omega S^2$

• To describe all faces $d_i$ systematically in terms of projections, consider the free group of rank $n$ in the following way. Let $\hat{F}_{n+1}$ be the quotient of the free group $F(z_0, z_1, \ldots, z_n)$ subject to the single relation $z_0 z_1 \cdots z_n = 1$. Let $\hat{z}_j$ be the image of $z_j$ in $\hat{F}_{n+1}$. The group $\hat{F}_{n+1}$ is written $\hat{F}(\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_n)$ in case the generators $\hat{z}_j$ are used.

• Clearly $\hat{F}_{n+1} \simeq F(\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_{n-1})$ is a free group of rank $n$. $\hat{F}_{n+1} = \pi_1(S^2 \setminus Q_{n+1})$.

• Define the faces $d_i$ and degeneracies $s_i$ on $\{\hat{F}_{n+1}\}_{n \geq 0}$ as follows:

$$
\begin{align*}
\hat{F} &= \{\hat{F}_{n+1}\}_{n \geq 0} \text{ with } d_i \text{ and } s_i \text{ defined as above is isomorphic to } F[S^1].
\end{align*}
$$
Combinatorial Description of $\pi_\ast(S^2)$

- **The Group** $G(n)$: generated by $x_1, x_2, \ldots, x_n$ subject to the relations:
  1) the ordered product of the generators $x_1 x_2 \cdots x_n = 1$ and
  2) the *iterated commutators* on the generators with the property that each generator occurs at least once in the commutator bracket.

- **Theorem.** (-) For each $n$, the homotopy group $\pi_n(S^2)$ is isomorphic to the center of $G(n)$.

- **Fixed-point Set Theorem.** (-) The Artin representation induces an action of the braid group $B_n$ on $G(n)$. Moreover the homotopy group $\pi_n(S^2)$ is isomorphic to the fixed set of the pure braid group action on $G(n)$.
Example

Let $\mathcal{G} = \{S_{n+1}\}_{n \geq 0}$ be the sequence of symmetric groups of degree $n + 1$. Then $\mathcal{G}$ is a simplicial set in the following way:

- $[n-1] \xrightarrow{d_i \cdot \sigma} [n]$, $d_i(\sigma)$
- $[n+1] \xrightarrow{s^i \cdot \sigma} [n]$, $s_i(\sigma)$
Crossed Simplicial Groups

- A **crossed simplicial group** is a simplicial set $G = \{G_n\}_{n \geq 0}$ for which each $G_n$ is a group, together with a group homomorphism $\mu : G_n \to S_{n+1}$, $g \mapsto \mu_g$ for each $n$, such that
  
  (i) $\mu$ is a simplicial map, and
  
  (ii) for $0 \leq i \leq n$, $d_i(gg') = d_i(g)d_i\cdot\mu_g(g')$ and $s_i(gg') = s_i(g)s_i\cdot\mu_g(g')$.

- An important example is that the sequence of Artin braid groups $\mathcal{B} = \{B_{n+1}\}_{n \geq 0}$ is a crossed simplicial group with the faces and degeneracies described as follows:

  Given an $(n + 1)$-strand braid $\beta \in B_{n+1}$, $d_i\beta$ is obtained by removing $(i + 1)$ st strand braid and $s_i\beta$ is obtained by doubling $(i + 1)$-strand for $0 \leq i \leq n$, where the stands are counted from initial points.

- Since the restriction of $d_i$ and $s_i$ to the pure braid groups are group homomorphisms, the sequence of groups $\mathcal{P} = \{P_{n+1}\}$ is a simplicial group.
Simplicial Structure on Configurations

- Let $(M, d)$ be a metric space with basepoint $w$ and let $\mathbb{R}^+$ denote $[0, \infty)$. A **steady flow** over $M$ is a (continuous) map $\theta: \mathbb{R}^+ \times M \to M$ such that
  1. for any $x \in M$, $\theta(0, x) = x$ and for $t > 0$
    
    $$0 < d(\theta(t, x), x) \leq t;$$
  2. $\theta|_{\mathbb{R}^+ \times \{w\}} : \mathbb{R}^+ \times \{w\} \to M$ is one-to-one;
  3. there exists a function $\epsilon : \mathbb{R}^+ \to (0, +\infty)$, $t \mapsto \epsilon_t$, such that
    
    $$\theta([0, \epsilon_t] \times \{\theta(t, w)\}) \subseteq \theta([t, \infty) \times \{w\})$$
    for any $t \in \mathbb{R}^+$.

- Let $M$ be a metric space with a steady flow. Then the sequence of groups $\mathcal{B}(M)^{\pi_1} = \{\pi_1(B(M, n + 1))\}_{n \geq 0}$ is a crossed simplicial group.

- Let $M$ be a metric space with a steady flow. Then the sequence of groups $\mathcal{F}(M)^{\pi_1} = \{\pi_1(F(M, n + 1))\}_{n \geq 0}$ is a simplicial group.
A \textbf{\(\Delta\)-group} \(G = \{G_n\}_{n \geq 0}\) consists of a \(\Delta\)-set \(G\) for which each \(G_n\) is a group and each \(d_i\) is a group homomorphism. The \textbf{Moore complex} \(NG = \{N_nG\}_{n \geq 0}\) of a \(\Delta\)-group \(G\) is defined by

\[N_nG = \bigcap_{i=1}^{n} \text{Ker}(d_i : G_n \rightarrow G_{n-1}).\]

Let \(G\) be a \(\Delta\)-group. Then \(d_0(N_nG) \subseteq N_{n-1}G\) and \(NG\) with \(d_0\) is a chain complex of groups.

Let \(G\) be a \(\Delta\)-group. An element in \(B_nG = d_0(N_{n+1}G)\) is called a Moore \textbf{boundary} and an element in \(Z_nG = \text{Ker}(d_0 : N_nG \rightarrow N_{n-1}G)\) is called a Moore \textbf{cycle}. The \(n\)th homotopy \(\pi_n(G)\) is defined to be the coset

\[\pi_n(G) = H_n(NG) = Z_nG/B_nG.\]
A crossed $\Delta$-group is a $\Delta$-set $G = \{G_n\}_{n \geq 0}$ for which each $G_n$ is a group, together with a group homomorphism $\mu : G_n \to S_{n+1}, \ g \mapsto \mu_g$ for each $n$, such that

(i) $\mu$ is a $\Delta$-map and
(ii) for $0 \leq i \leq n$, $d_i(gg') = d_i(g)d_i(\mu_g(g'))$.

A space $M$ is said to have a good basepoint $w_0$ if there is a continuous injection $\tilde{\theta} : \mathbb{R}^+ \to M$ with $\tilde{\theta}(0) = w_0$.

Let $M$ be a space with a good basepoint. Then

(1) $\mathcal{B}(M)^{\pi_1}$ is a crossed $\Delta$-group,
(2) $\mathcal{F}(M)^{\pi_1}$ is a $\Delta$-group,
Brunnian Braids

- An element in $\pi_1(B(M, n))$ is called a braid of $n$ strings over $M$.
- A pure braid of $n$ strings over $M$ means an element in $\pi_1(F(M, n))$.
- A braid is called Brunnian if it becomes a trivial braid when any one of its strings is removed.
- In the terminology of $\Delta$-groups, a braid

$$\beta \in \pi_1(B(M, n + 1)) = \mathcal{B}(M)^{\pi_1}$$

is Brunnian if and only if $d_i\beta = 1$ for $0 \leq i \leq n$. In other words, the Brunnian braids over $M$ are the Moore cycles in the $\Delta$-group $\mathcal{B}(M)^{\pi_1}$. The group of Brunnian braids of $n$ strings over $M$ is denoted by $\text{Brun}_n(M)$.

- Proposition. Let $\beta$ be a Brunnian braid of $n$ strings over a space $M$ with a good basepoint. If $n \geq 3$, then $\beta$ is a pure braid.
Theorems (JAMS paper with J. Berrick, F. Cohen and Y. L. Wong)

• There is an exact sequence of groups

\[ 1 \to \text{Brun}_{n+1}(S^2) \to \text{Brun}_n(D^2) \xrightarrow{f_*} \text{Brun}_n(S^2) \to \pi_{n-1}(S^2) \to 1 \]

for \( n \geq 5 \), where \( f_* \) is induced from the canonical embedding \( f : D^2 \to S^2 \).

• Let \( \mathcal{F}(S^2)^{\pi_1} \) be the \( \Delta \)-group defined above. Then, for each \( n \geq 1 \), the Moore homotopy group \( \pi_n(\mathcal{F}(S^2)^{\pi_1}) \) is a group, and there is an isomorphism of groups

\[ \pi_n(\mathcal{F}(S^2)^{\pi_1}) \cong \pi_n(S^2) \]

for \( n \geq 4 \).

• **Examples:** Since \( \pi_4(S^2) = \pi_5(S^2) = \mathbb{Z}/2 \) and \( \pi_6(S^2) = \mathbb{Z}/12 \), \( \text{Brun}_5(S^2) \equiv \mathbb{Z}/2 \mod \text{Brun}_5(D^2) \), \( \text{Brun}_6(S^2) \equiv \mathbb{Z}/2 \mod \text{Brun}_6(D^2) \) and \( \text{Brun}_7(S^2) \equiv \mathbb{Z}/12 \mod \text{Brun}_7(D^2) \).
Theorem (with Fred Cohen)

- Let $AP_* = \{P_{n+1}\}$ be the sequence of classical Artin pure braid groups with the simplicial structure given by deleting/doubling braids. Then $AP_*$ is a reduced simplicial group because $AP_0 = P_1 = 1$. Since $AP_1 = P_2 = \mathbb{Z}$, there is a unique simplicial homomorphism $\Theta : F[S^1] \to AP_*$ that sends the non-degenerate 1-simplex of $S^1$ to the generator of $AP_1 = \mathbb{Z}$, where $F[S^1]$ is Milnor’s $F[K]$-construction on $S^1$.

- The morphism of simplicial groups

$$\Theta : F[S^1] \to AP_*$$

is an embedding. Hence the homotopy groups of $F[S^1]$ are natural sub-quotients of $AP_*$, and the geometric realization of quotient simplicial set $AP_*/F[S^1]$ is homotopy equivalent to the 2-sphere. Furthermore, the image of $\Theta$ is the smallest simplicial subgroup of $AP_*$ which contains $A_{1,2}$. 
The $\Delta$-group $\mathbb{P}$

- Let $\delta : F(\mathbb{C}, n + 1) \to F(\mathbb{C}, n)$ be the map defined by
  \[
  \delta(z_1, z_2, \ldots, z_n) = \left(\frac{1}{z_2 - z_1}, \frac{1}{z_3 - z_1}, \ldots, \frac{1}{z_n - z_1}\right),
  \]
corresponding geometrically to the reflection map in $\mathbb{C}$ about the unit circle centered at $z_0$. Then $\delta$ induces a group homomorphism $\partial : P_{n+1} \to P_n$.

- **Proposition.** Let $\chi : B_n \to B_n$ be the mirror reflection, that is, $\chi$ is an automorphism of $B_n$ such that $\chi(\sigma_i) = \sigma_i^{-1}$ for all $i$. Let $\partial = \chi \circ \partial : P_{n+1} \to P_n$. Then $\mathbb{P} = \{P_n\}_{n \geq 0}$ is $\Delta$-group with $d_0 = \partial$ and $d_1, \ldots, d_n : P_n \to P_{n-1}$ are given by deleting strands.

- **Theorem.** The $\Delta$-group $\mathbb{P}$ is fibrant, and there is an isomorphism $\pi_n(\mathbb{P}) \cong \pi_n(S^2)$ for every $n$. 
Boundary Brunnian Braids

- The **boundary Brunnian braid** is defined to be
  \[ \text{Bd}_n = B_nP = \partial(\text{Brun}_{n+1})) = \tilde{\partial}(\text{Brun}_{n+1})) \]
  as the subgroup of \( P_n \).

- **Proposition.** \( \text{Bd}_n \) is a normal subgroup of \( B_n \).

- **Theorem.** The quotient groups \( P_n/\text{Bd}_n \) and \( B_n/\text{Bd}_n \) are finitely presented.

- **Questions:** Two questions then arise naturally:
  1) Is the group \( P_n/\text{Bd}_n \) torsion free?
  2) What is the center of \( B_n/\text{Bd}_n \)?

- The second question is a special case of the conjugation problem on braids. Namely, how to determine a braid \( \beta \in B_n \) such that the conjugation \( \sigma_i\beta\sigma_i^{-1} \) lies in the coset \( \beta\text{Bd}_n \) for each \( 1 \leq i \leq n - 1 \).
Theorem (New with Jingyan Li)

• For a subgroup $H$ of $G$, let

$$\sqrt{(H, G)} = \{ x \in G \mid x^q \in H \text{ for some } q \in \mathbb{Z} \}$$

denote the set of the roots of $H$ in $G$. Then

$$\sqrt{(Bd_n, P_n)/Bd_n}$$

is the set of torsion elements in $P_n/Bd_n$.

• Let $n \geq 4$.

1) $$\sqrt{(Bd_n, P_n)/Bd_n} \cong \pi_n(S^2).$$

2) There are isomorphisms of groups

$$\text{Center}(P_n/Bd_n) \cong \pi_n(S^2) \times \mathbb{Z}$$

$$\text{Center}(B_n/Bd_n) \cong \{ \alpha \in \pi_n(S^2) \mid 2\alpha = 0 \} \times \mathbb{Z}.$$

• This result gives a connection between the general higher homotopy groups of $S^2$ and the (special cases) of the conjugation problem on braids.
Mirror Reflection Theorem (New with Jingyan Li)

- By moving our steps to the next, the mirror reflection \( \chi : B_n \to B_n \) is a canonical operation on braids. Given a subgroup \( G \) of \( B_n \), one may ask whether there is a mirror symmetric braid \( \beta \) subject to \( G \), that is, the braids \( \beta \) satisfying the equation of cosets \( \chi(\beta)G = \beta G \). The answer to this question becomes very nontrivial. In the case \( G = B_d n \), the answer is again given in term of the homotopy group \( \pi_n(S^2) \).

- Let \( \text{Fix}^\phi(G) \) denote the subgroup of the fixed-point of an automorphism \( \phi \) of a group \( G \).

- **Theorem.** The subgroup \( B_d n \) is invariant under the mirror reflection \( \chi \). Moreover there is an isomorphism of groups

\[
\text{Fix}^\chi(B_n/B_d n) \cong \pi_n(S^2)
\]

for \( n \geq 3 \).