SUSPENSION SPLITTINGS AND HOPF INVARIANTS FOR RETRACTS OF THE LOOPS ON CO-H-SPACES

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ABSTRACT. James constructed a functorial homotopy decomposition $\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$ for path-connected, pointed CW-complexes X. We generalize this to a functorial decomposition of ΣA where A is any functorial retract of a looped co-H space. This is used to construct Hopf invariants in a more general context. As well, when $A = \Omega Y$ is the loops on a co-H space, we show that the wedge summands of $\Sigma\Omega Y$ further functorially decompose by using an action of an appropriate symmetric group.

1. Introduction

It is a classical result of James [5] that if X is a path-connected, pointed CW-complex then there is a functorial homotopy equivalence $\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$, where $X^{(n)}$ is the n-fold smash product of X with itself. After localizing at a prime p, Cohen, Selick and Wu [2, 11] refined James' decomposition by using the modular representation theory of the symmetric group Σ_n on n-letters to produce functorial wedge decompositions of $\Sigma X^{(n)}$.

The purpose of this paper is to extend James' decomposition to $\Sigma\Omega Y$, where Y is a co-H space, and to extend Cohen, Selick and Wu's decompositions to the wedge summands of $\Sigma\Omega Y$. The case when C is homotopy coassociative is known [12]. However, most interesting and useful co-H spaces are either not homotopy coassocative or not known to be, as this is a difficult property to check. The results in this paper therefore apply to a much wider class of spaces.

It is natural to try to extend James' result for $\Sigma\Omega\Sigma X$ to $\Sigma\Omega C$ for homological reasons. Take homology with coefficients in a field $\mathbf{k}$. Let $V = \widetilde{H}_*(X)$. By the Bott-Samelson Theorem, $H_*(\Omega\Sigma X) \cong T(V)$, where $T(V)$ is the free tensor algebra generated by V. As a module, $T(V) \cong \oplus_{n=1}^{\infty} V^{\otimes n}$, where $V^{\otimes n}$ is the n-fold tensor product of V with itself. Observe that $V^{\otimes n} \cong \widetilde{H}_*(X^{(n)})$. So there is a module isomorphism $H_*(\Omega\Sigma X) \cong \oplus_{n=1}^{\infty} \widetilde{H}_*(X^{(n)})$. James' splitting geometrically realizes this homology decomposition after one suspension. The homological situation generalizes. Given a graded module M, let $\Sigma^{-1}M$ be its desuspension. When Y is a co-H space it is well known that $H_*(\Omega Y) \cong T(\Sigma^{-1}\widetilde{H}_*(C))$. Thus there is a module decomposition $H_*(\Omega C) \cong \oplus_{n=1}^{\infty} \Sigma^{-(n-1)} \widetilde{H}_*(C)^{\otimes n}$. In Theorem 1.1 we show that this module decomposition can be geometrically realized after one suspension. This theorem is valid either integrally or p-locally.

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Theorem 1.1. *Let Y be a simply-connected co-H space. Then there is a functorial homotopy decomposition*

$$\Omega Y \simeq \bigvee_{n=1}^{\infty} [\Sigma\Omega Y]_n$$

where, for $n \geq 1$, $[\Sigma\Omega Y]_n$ is a space with the property that

$$\widetilde{H}_*([\Sigma\Omega Y]_n) \cong \Sigma^{-(n-1)}\widetilde{H}_*(Y^{\otimes n}).$$

For example, let p be an odd prime and let $S^{2p} \xrightarrow{\alpha_1} S^3$ represent the generator of $\pi_{2p}(S^3) = \mathbb{Z}/p\mathbb{Z}$, which is the least nonvanishing p-torsion homotopy group of S^3. Let Y be its homotopy cofiber. By [1], Y is a co-H space which is not homotopy coassociative. Therefore the results of [12] do not apply and so the decomposition of $\Sigma\Omega Y$ in Theorem 1.1 is new (the second author apologizes for incorrectly stating in [12] that such a decomposition was impossible). More generally, any retract Y of a suspension ΣX is a co-H space, and so Theorem 1.1 can be applied to Y. Numerous examples of retracts of suspensions exist, often in the form of wedge summands of homotopy decompositions. One example of wide interest is the p-local decomposition $\Sigma\mathbb{C}P^n \simeq \bigvee_{i=1}^{p-1} Y_i$, where $H_*(Y_i)$ consists of those elements in $H_*(\Sigma\mathbb{C}P^n)$ in degrees of the form $2i+1+2j(p-1)$ for some $j \geq 0$. Each space Y_i is a retract of $\Sigma\mathbb{C}P^n$ and so is a co-H space.

Localizing at a prime p, Theorem 1.1 has a significant generalization. Take homology with mod-p coefficients, and assume $\mathbb{Z}/p$ is the ground ring for any algebraic objects. Start with the fact that if Y is a co-H space then $H_*(\Omega Y) \cong T(V)$ where $V = \Sigma^{-1}\widetilde{H}_*(Y)$. In [8] it was shown that any functorial coalgebra retract $A(V)$ of $T(V)$ has a geometric realization. That is, there is a functorial retract $\bar{A}$ of ΩY with the property that $H_*(\bar{A}) \cong A(V)$. As well, the algebraic functor A can be refined degree-wise to functors A_n capturing the degree n elements in $A(V)$ for a particular V. The first part of the following theorem states the generalized decomposition result, while the second part states as a consequence that $A_n(V)$ can be geometrically realized after one suspension.

Theorem 1.2. *Let $A(V)$ be any functorial coalgebra retract of $T(V)$ and let $\bar{A}$ be the geometric realization of A. Then for any p-local simply connected co-H space Y of finite type and any p-local path-connected co-H-space Z, there is a functorial homotopy decomposition*

$$Z \wedge \bar{A}(Y) \simeq \bigvee_{n=1}^{\infty} [Z \wedge \bar{A}(Y)]_n$$

where, for $n \geq 1$, $[Z \wedge \bar{A}(Y)]$ is a space with the property that

$$\widetilde{H}_*([Z \wedge \bar{A}(Y)]_n) \cong \widetilde{H}_*(Z) \otimes A_n(\Sigma^{-1}\widetilde{H}_*(Y)).$$

In particular, for a p-local simply connected co-H-space Y there is a functorial homotopy decomposition

$$\Sigma\bar{A}(Y) \simeq \bigvee_{n=1}^{\infty} \bar{A}_n(Y)$$

such that

$$\Sigma^{-1}\bar{H}_*(\widetilde{A}_n(Y)) \cong A_n(\Sigma^{-1}\widetilde{H}_*(Y))$$

for each $n \geq 1$.

Observe that Theorem 1.1 follows by taking $\bar{A} = \Omega Y$ and $Z = S^1$. Another interesting consequence of Theorem 1.2 is to show that the smash product of two co-H spaces is a suspension.

Corollary 1.3. *Let Z be any p-local path-connected co-H space of finite type and let Y be any path-connected co-H-space of finite type. Then $Z \wedge Y$ is the suspension of a co-H space.*

To motivate the next result, observe that the symmetric group Σ_n on n letters acts on $X^{(n)}$ by permuting the smash factors. This induces a corresponding action of Σ_n on $\widetilde{H}_*(X)^{\otimes n}$ given by permuting the tensor factors. Because we can add in $\widetilde{H}_*(X)^{\otimes n}$, the action of Σ_n can be extended to an action of the group ring $\mathbb{Z}/p\mathbb{Z}[\Sigma_n]$. No such addition necessarily exists on the space level for $X^{(n)}$, but one does after suspending. That is, there is an action of $\mathbb{Z}_{(p)}[\Sigma_n]$ on $\Sigma X^{(n)}$ which, in homology, reduces to the suspension of the action of $\mathbb{Z}/p\mathbb{Z}[\Sigma_n]$ on $\widetilde{H}_*(X)^{\otimes n}$. Idempotents in $\mathbb{Z}_{(p)}[\Sigma_n]$ can be used to obtain decompositions of $\widetilde{H}_*(X)^{\otimes n}$ which can be geometrically realized after suspending. In general, a collection of idempotents $e_1, \ldots, e_k$ is mutually orthogonal if $e_1 + \cdots + e_k = 1$ and $e_i e_j = 0$ whenever $i \neq j$. Given such a collection of idempotents, one obtains corresponding maps $e_i \colon \Sigma X^{(n)} \longrightarrow \Sigma X^{(n)}$ with the property that, in homology, $(e_1)_*, \ldots, (e_k)_*$ are mutually orthogonal idempotents and so there is a decomposition $H_*(\Sigma X^{(n)}) \cong \oplus_{i=1}^k M_i$ where $M_i = \mathrm{Im}(e_i)_*$. On the space level, let T_i be the telescope of the map $e_i \colon \Sigma X^{(n)} \longrightarrow \Sigma X^{(n)}$ and consider the composite

$$f \colon \Sigma X^{(n)} \longrightarrow \Sigma X^{(n)} \vee \cdots \vee \Sigma X^{(n)} \longrightarrow T_1 \vee \cdots \vee T_k,$$

where the left map is given by the co-H structure and the right map is the wedge of maps to the telescopes. Since $H_*(T_i) \cong M_i$, f induces an isomorphism in homology and so is a homotopy equivalence. Moreover, as the action of $\mathbb{Z}_{(p)}[\Sigma_n]$ on $\Sigma X^{(n)}$ is natural, the decomposition of $\Sigma X^{(n)}$ is natural.

Thus to each collection of mutually orthogonal idempotents in $\mathbb{Z}_{(p)}[\Sigma_n]$, there is a corresponding natural wedge decomposition of $\Sigma X^{(n)}$. Our next result generalizes this to a natural wedge decomposition of the space $[\Sigma\Omega Y]_n$ in Theorem 1.1.

Theorem 1.4. *Let Y be a simply-connected p-local co-H space. Let $e_1, \ldots, e_k$ be a collection of mutually orthogonal idempotents in $\mathbb{Z}_{(p)}[\Sigma_n]$. Then there are maps $e_i \colon [\Sigma\Omega Y]_n \longrightarrow [\Sigma\Omega Y]_n$ such that $(e_1)_*, \ldots, (e_k)_*$ are mutually orthogonal idempotents, and there is a functorial homotopy decomposition*

$$[\Sigma\Omega Y]_n \simeq \bigvee_{i=1}^{k} T_i$$

where T_i is the telescope of e_i.

For example, the element $t = \Sigma_{\sigma \in \Sigma_k} \sigma \in \mathbb{Z}_{(p)}[\Sigma_k]$ has the property that $t \circ t = k!t$. So if $k < p$ then $k!$ is invertible and the element $\bar{t} = \frac{1}{k!}t$ is an idempotent. The idempotents $\bar{t}$ and $1 - \bar{t}$ are mutually orthogonal, so if Y is a co-H space there is a homotopy decomposition $[\Sigma\Omega Y]_k \simeq T_1 \vee T_2$ where T_1 is the telescope of $\bar{t}$ and T_2 is the telescope of $1 - \bar{t}$. The space T_1 has the property that $H_*(T_1)$ is isomorphic to the suspension of the submodule of length k symmetric tensors in $(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes k}$.

Our second purpose in this paper is to define and study Hopf invariants for ΩY when Y is a simply-connected p-local co-H space. Given the decomposition

$$\Sigma\bar{A}(Y)\simeq\bigvee_{n=1}^{\infty}\bar{A}_n(Y)$$

of Theorem 1.2 we obtain Hopf invariants

$$\bar{A}(Y)\longrightarrow\Omega(\bar{A}_n(Y)).$$

For computational purpose in homology, it is useful to make a particular choice of these Hopf invariants. To do this, we consider analogues of the combinatorial James-Hopf invariants in [5] and restrict to the case when $A(Y)=\Omega Y$. For a path-connected space X, let

$$H_n\colon\Omega\Sigma X\longrightarrow\Omega\Sigma X^{(n)}$$

be the n^{th}-combinatorial James-Hopf invariant and let

$$\bar{H}_n\colon\Sigma\Omega\Sigma X\longrightarrow\Sigma X^{(n)}$$

be its adjoint. By [4], there is a map $s\colon Y\longrightarrow\Sigma\Omega Y$ which is a right homotopy inverse of the evaluation map. Let $\bar{H}^Y_n$ be the composite

$$\bar{H}^Y_n:\Sigma\Omega Y\xrightarrow{\Sigma\Omega s}\Sigma\Omega\Sigma\Omega Y\xrightarrow{\bar{H}_n}\Sigma(\Omega Y)^{(n)}\longrightarrow[\Sigma\Omega Y]_n$$

where the right map is the one to the homotopy colimit. Define the n-th Hopf invariant

$$H^Y_n:\Omega Y\longrightarrow\Omega[\Sigma\Omega Y]_n$$

as the adjoint of $\bar{H}^Y_n$.

On the algebraic side, if V is a graded $\mathbb{Z}/p$-module then in [10] an n^{th}-*algebraic James-Hopf invariant* $\mathcal{H}_n\colon T(V)\longrightarrow T(V^{\otimes n})$ was defined and shown to have the property that the James-Hopf invariant $\Omega\Sigma X\xrightarrow{H_n}\Omega\Sigma X^{(n)}$ satisfies $(H_n)_*=\mathcal{H}_n$. As the algebraic map exists for any tensor algebra, it can be applied to $H_*(\Omega Y)\cong T(\Sigma^{-1}\widetilde{H}_*(Y))$, and it is natural to ask whether $(H^Y_n)_*=\mathcal{H}_n$. The next theorem shows that this is true, at least after taking the associated graded corresponding to the augmentation ideal filtration on $H_*(\Omega Y)$.

Theorem 1.5. *Let Y be a simply-connected p-local co-H space. Then there is a commutative diagram*

$$\begin{array}{ccc}T(\Sigma^{-1}\bar{H}_*(Y)) & \xrightarrow{\mathcal{H}_n} & T((\Sigma^{-1}\bar{H}_*(Y))^{\otimes n})\\ \downarrow\cong & & \downarrow\cong\\ E^0H_*(\Omega Y) & \xrightarrow{E^0H^Y_{n*}} & E^0H_*(\Omega[\Sigma\Omega Y]_n)\end{array}$$

where $\mathcal{H}_n\colon T(V)\longrightarrow T(V^{\otimes n})$ is the n^{th}-algebraic James-Hopf map.

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2. Preliminary facts about co-H spaces

This section briefly records some information about co-H spaces which will then be assumed throughout. All statements hold either integrally or p-locally.

If Y is a co-H space, Ganea [4] showed that there is a map $s\colon Y \longrightarrow \Sigma\Omega Y$ which is a right homotopy inverse of the evaluation map $\sigma\colon \Sigma\Omega Y \longrightarrow Y$.

In mod-p homology, there is an algebra isomorphism

$$H_*(\Omega Y) \cong T(\Sigma^{-1}\widetilde{H}_*(Y)). \tag{2.1}$$

The tensor algebra $T(\Sigma^{-1}\widetilde{H}_*(Y))$ can be given the structure of a Hopf algebra by declaring that the generators are primitive and then extending multiplicatively. It is important to note that the isomorphism in (2.1) may *not* be as Hopf algebras. In general, this is true only if $Y = \Sigma^2 X$ for some space X. However, filtering $H_*(\Omega Y)$ by the augmentation ideal filtration, there is an isomorphism of Hopf algebras

$$E^0 H_*(\Omega Y) \cong T(\Sigma^{-1}\widetilde{H}_*(Y)).$$

Throughout much of the paper we will use this associated graded object as it allows us to calculate as if Y were a double suspension. In particular, the map $Y \xrightarrow{s} \Sigma\Omega Y$ has the property that

$$H_*(Y) \xrightarrow{s_*} E^0 H_*(\Sigma\Omega Y) \cong \Sigma E^0 T(\Sigma^{-1}\widetilde{H}_*(Y))$$

is the suspension of the inclusion of the generating set, and the evaluation map $\Sigma\Omega Y \xrightarrow{\sigma} Y$ has the property that

$$E^0 H_*(\Sigma\Omega Y) \cong \Sigma E^0 T(\Sigma^{-1}\widetilde{H}_*(Y)) \xrightarrow{\sigma_*} H_*(Y)$$

is the suspension of the projection onto the generating set.

3. Suspension Splitting Theorems

In this section we are going to prove Theorems 1.1 and 1.4. We begin with a general splitting lemma.

A *graded space* means a space W with a homotopy decomposition

$$\phi\colon W \xrightarrow{\simeq} \bigvee_{n=1}^{\infty} W_n.$$

For any graded space W, the homology $\widetilde{H}_*(W)$ is filtered by

$$I^t \widetilde{H}_*(W) = \phi_*^{-1}(\widetilde{H}_*(\bigvee_{n=t}^{\infty} W_n))$$

for $t \geq 1$. A *graded co-H space* means a graded space W such that W is a co-H space. As a retract of a co-H space is a co-H space, each summand W_n is also a co-H space. The following lemma gives a general criterion for decomposing retracts of graded co-H spaces in term of the grading factors.

Lemma 3.1. *Let W be a simply-connected p-local graded co-H space of finite type. Let $f\colon W \longrightarrow W$ be a self-map such that in mod-p homology:*

1) *$f_*\colon \widetilde{H}_*(W) \longrightarrow \widetilde{H}_*(W)$ preserves the filtration;*
2) *the induced bigraded map $E^0 f_*\colon E^0 H_*(W) \longrightarrow E^0 H_*(W)$ is an idempotent.*

Let $A(f) = \operatorname{hocolim}_f W$ *be the homotopy colimit and let* $A_n(f) = \operatorname{hocolim}_{g_n} W_n$*, where* g_n *is the composite*

$$g_n\colon W_n \hookrightarrow \bigvee_{k=1}^{\infty} W_k \xrightarrow{\phi^{-1}} W \xrightarrow{f} W \xrightarrow{\phi} \bigvee_{k=1}^{\infty} W_k \twoheadrightarrow W_n.$$

Then there is a homotopy decomposition

$$A(f) \simeq \bigvee_{n=1}^{\infty} A_n(f)$$

such that $\widetilde{H}_*(A_n(f)) \cong \operatorname{Im}(g_n)_*$*. This homotopy decomposition is natural if both* ϕ *and* f *are.*

Proof. Let

$$g = \phi \circ f \circ \phi^{-1}\colon \bigvee_{k=1}^{\infty} W_k \longrightarrow \bigvee_{k=1}^{\infty} W_k.$$

By definition of g there is a homotopy commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & W \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \bigvee_{k=1}^{\infty} W_k & \xrightarrow{g} & \bigvee_{k=1}^{\infty} W_k. \end{array} \tag{3.1}$$

We have $A(f) = \operatorname{hocolim}_f W$; let $B = \operatorname{hocolim}_g \bigvee_{k=1}^{\infty} W_k$. Taking homotopy colimits horizontally in (3.1) shows that ϕ induces a homotopy equivalence

$$\tilde{\phi}\colon A(f) = \operatorname{hocolim}_f W \xrightarrow{\simeq} B = \operatorname{hocolim}_g \bigvee_{k=1}^{\infty} W_k.$$

Now consider how (3.1) behaves in homology. By assumption, f_* preserves the filtration and so each submodule $\widetilde{H}_*(\bigvee_{k=n}^{\infty} W_k)$ is invariant under g_*. Thus if we filter $\widetilde{H}_*(\bigvee_{k=1}^{\infty} W_k)$ by $\widetilde{H}_*(\bigvee_{k=n}^{\infty} W_k)$ then $E_n^0 f_*\colon E_n^0 H_*(W) \cong \widetilde{H}_*(W_n) \longrightarrow E_n^0 H_*(W) \cong \widetilde{H}_*(W_n)$ equals $E_n^0 g_*$. In particular, by assumption $E^0 f_*$ is an idempotent, so $E^0 g_*$ is as well, and therefore so is each $E_n^0 g_*$ for $n \geq 1$. Now observe that the definition of g_n implies that $(g_n)_* = E_n^0 g_*$ and so $(g_n)_*$ is an idempotent. Therefore the composite

$$W_n \xrightarrow{\text{comult}} W_n \vee W_n \longrightarrow A_n(f) \vee \operatorname{hocolim}_{\mathrm{id}-g_n} W_n$$

induces an isomorphism in homology and so is a homotopy equivalence. Hence the map to the colimit $W_n \longrightarrow A_n(f)$ admits a cross-section $s_n\colon A_n(f) \longrightarrow W_n$ with the property that

$$\operatorname{Im}(s_{n*}\colon \widetilde{H}_*(A_n(f)) \longrightarrow \widetilde{H}_*(W_n)) = \operatorname{Im}(g_n)_*.$$

Consequently, $\widetilde{H}_*(A_n f) \cong \operatorname{Im}(g_n)_*$, as asserted.

It remains to prove that $A(f) \simeq \bigvee_{n=1}^{\infty} A_n(f)$. Consider the composite

$$\theta\colon \bigvee_{n=1}^{\infty} A_n(f) \xrightarrow{\bigvee_{n=1}^{\infty} s_n} \bigvee_{n=1}^{\infty} W_n \longrightarrow \operatorname{hocolim}_g \bigvee_{n=1}^{\infty} W_n = B.$$

Let

$$\widetilde{H}_*(\bigvee_{n=1}^{\infty} A_n(f)) = \bigoplus_{n=1}^{\infty} \widetilde{H}_*(A_n(f))$$

be filtered by $\bigoplus_{k=n}^{\infty} \widetilde{H}_*(A_k(f))$. Then

$$\theta_*\colon \widetilde{H}_*(\bigvee_{n=1}^{\infty} A_n(f)) \longrightarrow \widetilde{H}_*(B)$$

is filtration preserving and there is an isomorphism

$$E^0\theta_*\colon E^0\widetilde{H}_*(\bigvee_{n=1}^{\infty} A_n(f)) \xrightarrow{\cong} E^0\widetilde{H}_*(B).$$

Since W is of finite type, the filtrations on $\widetilde{H}_q(\bigvee_{n=1}^{\infty} A_n(f))$ and $\widetilde{H}_q(B)$ are finite dimensional for each q. Thus

$$\theta_*\colon \widetilde{H}_*(\bigvee_{n=1}^{\infty} A_n(f)) \longrightarrow \widetilde{H}_*(B)$$

is isomorphism and so θ is a homotopy equialence. Hence the composite

$$A(f) \xrightarrow{\tilde{\phi}} B \xrightarrow{\theta^{-1}} \bigvee_{n=1}^{\infty} A_n(f)$$

is a homotopy equivalence. □

Let X be a path-connected space. Let $H_*(\Omega\Sigma X)$ be filtered by the powers of the augmentation ideal filtration. From the classical suspension splitting Theorem of James [5], there is a homotopy equivalence

$$\phi\colon \Sigma\Omega\Sigma X \xrightarrow{\simeq} \bigvee_{n=1}^{\infty} \Sigma X^{(n)}.$$

This equivalence makes $\Sigma\Omega\Sigma X$ a simply-connected graded co-H space. Observe that the filtration

$$I^t\widetilde{H}_*(\Sigma\Omega\Sigma X) = \phi_*^{-1}(\widetilde{H}_*(\bigvee_{n=t}^{\infty} \Sigma X^{(n)}))$$

coincides with the suspension of the augmentation ideal filtration of $H_*(\Omega\Sigma X)$.

Let W and W' be graded spaces. Then $W \wedge W'$ is a graded space and there is a homotopy equivalence

$$W \wedge W' \xrightarrow{\phi_W \wedge \phi_{W'}} (\bigvee_{n=1}^{\infty} W_n) \wedge (\bigvee_{n=1}^{\infty} W'_n) = \bigvee_{n=1}^{\infty} \bigvee_{i=1}^{n-1} W_i \wedge W'_{n-i}.$$

Further, $\widetilde{H}_*(W \wedge W')$ is isomorphic to the tensor product $\widetilde{H}_*(W) \otimes \widetilde{H}_*(W')$ as filtered modules. Also, if W is a graded co-H space, then $W \wedge W'$ is a graded co-H space.

Theorem 3.2. *Let Z be a p-local path connected co-H space of finite type. Then the following hold:*

(1) *for any p-local path connected space X of finite type, there is a homotopy decomposition*

$$Z \wedge \Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} Z \wedge X^{(n)};$$

(2) *for any p-local simply connected co-H space Y of finite type, there is a homotopy decomposition*

$$Z \wedge \Omega Y \simeq \bigvee_{n=1}^{\infty} [Z \wedge \Omega Y]_n$$

where, for $n \geq 1$, $[Z \wedge \Omega Y]_n$ is a space with the property that

$$\widetilde{H}_*([Z \wedge \Omega Y]_n) \cong \widetilde{H}_*(Z) \otimes (\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}.$$

Proof. (1) Since Z is a co-H space, there is a map $s\colon Z \longrightarrow \Sigma\Omega Z$ which is a right homotopy inverse of the evaluation map $\sigma\colon \Sigma\Omega Z \longrightarrow Z$. Let f be the composite

$$f\colon \Sigma\Omega Z \wedge \Omega\Sigma X \xrightarrow{\sigma\wedge \mathrm{id}_{\Omega\Sigma X}} Z \wedge \Omega\Sigma X \xrightarrow{s\wedge \mathrm{id}_{\Omega\Sigma X}} \Sigma\Omega Z \wedge \Omega\Sigma X.$$

We aim to apply Lemma 3.1. Let $W = \Sigma\Omega Z \wedge \Omega\Sigma X$ and consider the self-map $W \xrightarrow{f} W$. The James equivalence $\Sigma\Omega\Sigma X \xrightarrow{\phi} \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$ induces a homotopy equivalence $\varphi\colon W \xrightarrow{\simeq} \bigvee_{n=1}^{\infty} \Sigma\Omega Z \wedge X^{(n)}$ which gives W the structure of a simply-connected graded co-H space. Since ϕ_* is filtration preserving, the induced filtration on $\widetilde{H}_*(W)$ coincides with tensoring the augmentation ideal filtration on $\widetilde{H}_*(\Omega\Sigma X)$ with $H_*(\Sigma\Omega Z)$. Thus f_* is filtration-preserving. Since s is a right homotopy inverse of σ, the composite $f \circ f$ is homotopic to f. Therefore f is an idempotent and so $E^0 f_*$ is an idempotent. Hence the hypotheses of Lemma 3.1 are satisfied. To state the conclusion, observe that $A(f) = \mathrm{hocolim}_f\, W = Z \wedge \Omega\Sigma X$; the map g_n is the composite

$$\Sigma\Omega Z \wedge X^{(n)} \hookrightarrow \bigvee_{k=1}^{\infty} \Sigma\Omega Z \wedge X^{(k)} \xrightarrow{\varphi^{-1}} W \xrightarrow{f} W \xrightarrow{\varphi} \bigvee_{k=1}^{\infty} \Sigma\Omega Z \wedge X^{(k)} \twoheadrightarrow \Sigma\Omega Z \wedge X^{(n)};$$

and as the image of f_* is $\widetilde{H}_*(Z \wedge \Omega\Sigma X)$ the fact that f_* preserves the filtration induced by the augmentation ideal filtration on $\widetilde{H}_*(\Omega\Sigma X)$ implies that the image of $(g_n)_*$ is isomorphic to $\widetilde{H}_*(Z \wedge X^{(n)})$. Thus $A_n(f) = \mathrm{hocolim}_{g_n}\, \Sigma\Omega Z \wedge X^{(n)}$ has reduced homology isomorphic to $\widetilde{H}_*(Z \wedge X^{(n)})$. Moreover, the same filtration reasoning implies that the composite $Z \wedge X^{(n)} \xrightarrow{s\wedge 1} \Sigma\Omega Z \wedge X^{(n)} \xrightarrow{g_n} \Sigma\Omega Z \wedge X^{(n)}$ has the property that $\mathrm{Im}(g_n \circ (s\wedge 1))_* \cong \mathrm{Im}(g_n)_*$. Thus the composite $Z \wedge X^{(n)} \xrightarrow{s\wedge 1} \Sigma\Omega Z \wedge X^{(n)} \longrightarrow A_n(f)$ induces an isomorphism in homology and so is a homotopy equivalence. Consequently, the homotopy decomposition $A(f) \simeq \bigvee_{n=1}^{\infty} A_n(f)$ of Lemma 3.1 becomes, in this case, $Z \wedge \Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} Z \wedge X^{(n)}$.

(2) Since Y is a co-H space, there is a map $s\colon Y \longrightarrow \Sigma\Omega Y$ which is a right homotopy inverse of the evaluation map $\sigma\colon \Sigma\Omega Y \longrightarrow Y$. Let $\tilde{f}$ be the composite

$$\tilde{f}\colon \Omega\Sigma\Omega Y \xrightarrow{\Omega\sigma} \Omega Y \xrightarrow{\Omega s} \Omega\Sigma\Omega Y.$$

Observe that $\tilde{f}_*\colon H_*(\Omega\Sigma\Omega Y) \longrightarrow H_*(\Omega\Sigma\Omega Y)$ preserves the augmentation ideal filtration because $f_* = (\Omega(s\circ\sigma))_*$ is an algebra map. Also, $\tilde{f}$ is an idempotent because s is a right homotopy inverse of σ.

Now consider the map

$$f\colon Z\wedge\Omega\Sigma\Omega Y\xrightarrow{1\wedge\tilde{f}}Z\wedge\Omega\Sigma\Omega Y.$$

We aim to apply Lemma 3.1 to $W=Z\wedge\Omega\Sigma\Omega Y$ and the self-map f. By part (1), there is a homotopy equivalence

$$\phi\colon W=Z\wedge\Omega\Sigma\Omega Y\longrightarrow\bigvee_{n=1}^{\infty}Z\wedge(\Omega Y)^{(n)}.$$

This gives W the structure of a simply-connected graded co-H space. The induced filtration on $\widetilde{H}_*(W)$ coincides with tensoring the augmentation ideal filtration on $\widetilde{H}_*(\Omega\Sigma\Omega Y)$ with $H_*(Z)$. Thus, since $\tilde{f}_*$ preserves the augmentation ideal filtration, f_* preserves the filtration on $\widetilde{H}_*(W)$. Since $\tilde{f}$ is an idempotent, so is f by its definition, and therefore so is E^0f_*. Thus the hypotheses of Lemma 3.1 are satisfied. To state the conclusion, observe that $\mathrm{hocolim}_{\tilde{f}}\,\Omega\Sigma\Omega Y=\Omega Y$ and so $A(f)=\mathrm{hocolim}_f\,Z\wedge\Omega\Sigma\Omega Y=Z\wedge\Omega Y$. The map g_n is the composite

$$Z\wedge(\Omega Y)^{(n)}\hookrightarrow\bigvee_{k=1}^{\infty}Z\wedge(\Omega Y)^{(k)}\xrightarrow{\phi^{-1}}W\xrightarrow{f}W\xrightarrow{\phi}\bigvee_{k=1}^{\infty}Z\wedge(\Omega Y)^{(k)}\twoheadrightarrow Z\wedge(\Omega Y)^{(n)}.$$

Since there is a Hopf algebra isomorphism

$$\mathrm{Im}(E^0f_*)=E^0H_*(W)=H_*(Z\wedge\Omega Y)\cong H_*(Z)\otimes E^0T(\Sigma^{-1}\widetilde{H}_*(Y)),$$

the fact that f_* preserves the augmentation ideal filtration implies that the image of $E^0(g_n)_*$ is isomorphic to $\widetilde{H}_*(Z)\otimes(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$. Thus $A_n(f)=\mathrm{hocolim}_{g_n}\,Z\wedge(\Omega Y)^{(n)}$ satisfies

$$\widetilde{H}_*(A_n(f))\cong\mathrm{Im}(g_n)_*\cong\widetilde{H}_*(Z)\otimes(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}.$$

Let $[Z\wedge\Omega Y]_n=A_n(f)$. Then the homotopy decomposition $A(f)\simeq\bigvee_{n=1}^{\infty}A_n(f)$ of Lemma 3.1 becomes, in this case, $Z\wedge\Omega Y\simeq\bigvee_{n=1}^{\infty}[Z\wedge\Omega Y]_n$. □

As a special case of Theorem 3.2 we obtain the following.

Proof of Theorem 1.1: Take $Z=S^1$ in Theorem 3.2. □

Remark 3.3. *By inspecting the proof, Theorem 3.2 also holds for integral co-H spaces.*

Another application of Theorem 3.2 is to show that the smash of two co-H spaces is a suspension.

Corollary 3.4. *Let Z be any p-local path-connected co-H space of finite type and let Y be any path-connected co-H-space of finite type. Then $Z\wedge Y$ is the suspension of a co-H space.*

Proof. If Y is simply connected, then the assertion follows from the fact that the composite

$$\Sigma[Z\wedge\Omega Y]_1\hookrightarrow\Sigma Z\wedge\Omega Y\simeq Z\wedge\Sigma\Omega Y\xrightarrow{\mathrm{id}_Z\wedge\sigma}Z\wedge Y$$

is a homotopy equivalence as it induces an isomorphism on homology.

Let Y be any path-connected co-H-space. By [3], $\pi_1(Y)$ is a free group and so there is a map $f\colon\bigvee_{\alpha}S^1\longrightarrow Y$ such that

$$f_*\colon\pi_1(\bigvee_{\alpha}S^1)\longrightarrow\pi_1(Y)$$

is an isomorphism. Let Y' be the homotopy cofibre of the map f and let $j\colon Y \longrightarrow Y'$ be the map to the cofibre. Since f induces an isomorphism on π_1 and $\pi_1(Y \vee Y)$ is isomorphic to the free group $\pi_1(Y) * \pi_1(Y)$, $f \vee f$ induces an isomorphism on π_1 and there is a homotopy commutative diagram

$$\begin{array}{ccc} \bigvee_\alpha S^1 & \xrightarrow{\ f\ } & Y \\ \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu} \\ (\bigvee_\alpha S^1) \vee (\bigvee_\alpha S^1) & \xrightarrow{f \vee f} & Y \vee Y. \end{array}$$

Thus f induces a map $Y' \longrightarrow Y' \vee Y'$ of cofibres, giving Y' the structure of a co-H space. Also, since f induces an isomorphism on π_1, it also induces an isomorphism on the abelianizations on H_1. Thus Y' is simply-connected. Further, the dual isomorphism on H^1 implies implies that there is a map $p\colon Y \longrightarrow K(\pi_1(Y), 1) = \bigvee_\alpha S^1$ which is a left homotopy inverse of f. Thus the composite

$$\phi\colon Y \xrightarrow{\mu} Y \vee Y \xrightarrow{p \vee j} \bigvee_\alpha S^1 \vee Y'$$

is a homology isomorphism. Now smashing with Z gives a homology isomorphism

$$Z \wedge Y \simeq (Z \wedge (\bigvee_\alpha S^1)) \vee (Z \wedge Y') = (\bigvee_\alpha \Sigma Z) \vee (Z \wedge Y').$$

Observe that each of $Z \wedge Y$, ΣZ, and $Z \wedge Y'$ is simply-connected. Thus this homology isomorphism is a homotopy equivalence. Since Y' is a simply-connected co-H space, the first part of the proof shows that $Z \wedge Y' = \Sigma[Z \wedge \Omega Y']_1$ where $[Z \wedge \Omega Y']_1$ is a co-H space. Hence $Z \wedge Y$ is homotopy equivalent to the suspension of the co-H space $(\bigvee_\alpha Z) \vee [Z \wedge \Omega Y']_1$. □

Next, we use the construction of $[\Sigma\Omega Y]_n$ to prove Theorem 1.4.

Proof of Theorem 1.4: We are given a simply-connected p-local co-H space Y and a collection $e_1, \ldots, e_k$ of mutually orthogonal idempotents in $\mathbb{Z}_{(p)}[\Sigma_n]$. As described in the Introduction, these idempotents give rise to maps $e_k\colon \Sigma(\Omega Y)^{(n)} \longrightarrow \Sigma(\Omega Y)^{(n)}$ with the property that, in homology, $(e_1)_*, \ldots, (e_k)_*$ are mutually orthogonal idempotents, implying that there is a decomposition $H_*(\Sigma(\Omega Y)^{(n)}) \cong \oplus_{i=1}^k M_i$ where $M_i = \mathrm{Im}(e_i)_*$.

As in the proof of Theorem 3.2, the space $[\Sigma\Omega Y]_n$ is constructed as a retract of $\Sigma(\Omega Y)^{(n)}$ with the property that $E^0\widetilde{H}_*([\Sigma\Omega Y]_n) \longrightarrow E^0\widetilde{H}_*(\Sigma(\Omega Y)^{(n)}) \longrightarrow E^0\widetilde{H}_*([\Sigma\Omega Y]_n)$ is identified with the suspension of the inclusion and projection $(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n} \longrightarrow T(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n} \longrightarrow (\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$. Consider the composites

$$\bar{e}_i\colon [\Sigma\Omega Y]_n \longrightarrow \Sigma(\Omega Y)^{(n)} \xrightarrow{e_i} \Sigma(\Omega Y)^{(n)} \longrightarrow [\Sigma\Omega Y]_n.$$

Since the idempotent $(e_i)_*$ is natural, the fact that the first and last maps in the definition of $\bar{e}_i$ are the suspensions of the inclusion and projection on E^0H_* implies that $(\bar{e}_i)_*$ is an idempotent. By similar reasoning, $(\bar{e}_1)_*, \ldots, (\bar{e}_k)_*$ are a mutually orthogonal set of idempotents. Thus there is a homotopy decomposition $[\Sigma\Omega Y]_n \simeq \bigvee_{i=1}^k T_i$ where $T_i = \mathrm{hocolim}_{\bar{e}_i}[\Sigma\Omega Y]_n$, as claimed.

Finally, the naturality of this decomposition follows from the naturality of the retraction of $[\Sigma\Omega Y]_n$ off $\Sigma(\Omega Y)^{(n)}$ and the naturality of the idempotents e_i. □

4. Generalization to functorial retracts of looped co-H spaces

This section generalizes the decomposition of $\Sigma\Omega Y$ in Theorem 1.1 to the p-local decomposition of a functorial retract of $\Sigma\Omega Y$ in Theorem 1.2. To begin, we state the relationship between functorial coalgebra decompositions of tensor algebras and functorial decompositions of looped co-H spaces, proved in [8]. Let **CoH** be the category of simply-connected p-local co-H spaces and co-H maps.

Theorem 4.1 (Geometric Realization Theorem)**.** *Let Y be any simply connected co-H-space of finite type and let*

$$T(V) \cong A(V) \otimes B(V)$$

any functorial coalgebra decomposition for ungraded modules over $\mathbb{Z}/p$. Then there exist homotopy functors $\bar{A}$ and $\bar{B}$ from **CoH** *to spaces such that*

(1) *there is a functorial decomposition*

$$\Omega Y \simeq \bar{A}(Y) \times \bar{B}(Y);$$

(2) *in mod p homology the decomposition*

$$H_*(\Omega Y) \cong H_*(\bar{A}(Y)) \otimes H_*(\bar{B}(Y))$$

is with respect to the augmentation ideal filtration;

(3) *in mod p homology*

$$E^0 H_*(\bar{A}(Y)) = A(\Sigma^{-1}\widetilde{H}_*(Y)) \quad \text{and} \quad E^0_H * (\bar{B}(Y)) = B(\Sigma^{-1}\widetilde{H}_*(Y)),$$

where A and B here are the canonical extensions of the ungraded functors to graded modules. □

One example of a functorial coalgebra decomposition of $T(V)$ of particular interest from [9] is $T(V) \cong A^{\min}(V) \otimes B^{\max}(V)$, where $A^{\min}(V)$ is the minimal functorial coalgebra retract of $T(V)$. This has a geometric realization as $\Omega Y \simeq \bar{A}^{\min}(Y) \times \bar{B}^{\max}(Y)$ where $\bar{A}^{\min}(Y)$ is the minimal functorial homotopy retract of ΩY.

Proof of Theorem 1.2: By [9], any functorial coalgebra retract A of the tensor algebra functor T is obtained as the image of an idempotent. That is, for any V, there is a natural coalgebra map $\alpha\colon T(V) \longrightarrow T(V)$ which is an idempotent and whose image is $A(V)$. By Theorem 4.1, if Y is a simply-connected p-local co-H space then $A(\Sigma\widetilde{H}_*(Y))$ is geometrically realized by a space $\bar{A}(Y)$. The proof of this in [8] uses the fact that the coalgebra map α induces a natural map $\bar{\alpha}\colon \Omega Y \longrightarrow \Omega Y$ such that $H_*(\Omega Y) \xrightarrow{\bar{\alpha}_*} H_*(\Omega Y)$ preserves the augmentation ideal filtration and $E^0\bar{\alpha}_* = \alpha$. The space $\bar{A}(Y)$ is then defined as $\operatorname{hocolim}_{\bar{\alpha}} \Omega Y$.

The assertions now follow from Lemma 3.1 and Theorem 3.2 by taking $W = Z \wedge \Omega Y$ and defining the self-map $f\colon W \longrightarrow W$ as the composite

$$Z \wedge \bar{A}(Y) \longrightarrow Z \wedge \Omega Y \xrightarrow{1\wedge\bar{\alpha}} Z \wedge \Omega Y \longrightarrow Z \wedge \bar{A}(Y).$$

□

5. Hopf Invariants

In this section we prove Theorem 1.5. As the construction and computation of the James-Hopf invariants for ΩY rely on the known James-Hopf invariants for $\Omega\Sigma X$, we begin with some information for the known case.

Let

$$H\colon \Omega\Sigma X \longrightarrow \Omega\Sigma(\bigvee_{n=1}^{\infty} X^{(n)})$$

be the combinatorial James-Hopf map, and let

$$H_n\colon \Omega\Sigma X \longrightarrow \Omega\Sigma X^{(n)}$$

be the map obtained by pinching to the n^{th}-wedge summand. Let

$$\bar{H}\colon \Sigma\Omega\Sigma X \simeq \Sigma J(X) \xrightarrow{\simeq} \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$$

$$\bar{H}_n\colon \Sigma\Omega\Sigma X \longrightarrow \Sigma X^{(n)}$$

be the adjoints of H and H_n respectively. The homotopy equivalence $\bar{H}$ gives a particular choice of a graded co-H space structure on $\Sigma\Omega\Sigma X$. This choice respects the natural filtrations in homology. That is, let $H_*(\Omega\Sigma X)$ be filtered by the products of the augmentation ideal and let $H_*(\bigvee_{n=1}^{\infty} X^{(n)})$ be filtered by

$$\bigoplus_{t\geq n} H_*(X^{(t)}).$$

Then by [10, Proposition 3.7], the isomorphism

$$\bar{H}_*\colon H_*(\Sigma\Omega\Sigma X) \longrightarrow H_*(\bigvee_{n=1}^{\infty} \Sigma X^{(n)})$$

preserves the filtration.

Now suppose Y is a simply-connected p-local co-H space of finite type with cross-section $s\colon Y \longrightarrow \Sigma\Omega Y$. By Theorem 3.2 (b) (with $Z = S^1$), there is a homotopy equivalence $\Sigma\Omega Y \simeq \bigvee_{n=1}^{\infty}[\Sigma\Omega Y]_n$. The proof of this depended on the existence of a (filtration preserving) homotopy decomposition $\Sigma\Omega\Sigma(\Omega Y) \simeq \bigvee_{n=1}^{\infty} \Sigma(\Omega Y)^{(n)}$. The homotopy equivalence $\bar{H}$ gives a particular choice of such a decomposition of $\Sigma\Omega\Sigma(\Omega Y)$, and therefore determines a particular choice of a decomposition of $\Sigma\Omega Y$. Specifically, let g_n be the composite

$$g_n\colon \Sigma(\Omega Y)^{(n)} \xrightarrow{i} \bigvee_{k=1}^{\infty} \Sigma(\Omega Y)^{(k)} \xrightarrow{\bar{H}^{-1}} \Sigma\Omega\Sigma\Omega Y \xrightarrow{\Sigma\Omega\sigma} \Sigma\Omega Y$$

$$\xrightarrow{\Sigma\Omega s} \Sigma\Omega\Sigma\Omega Y \xrightarrow{\bar{H}} \bigvee_{k=1}^{\infty} \Sigma(\Omega Y)^{(k)} \xrightarrow{\pi} \Sigma(\Omega Y)^{(n)}$$

where i is the inclusion and π is the pinch map. Then we can take $[\Sigma\Omega Y]_n = \mathrm{hocolim}_{g_n} \Sigma(\Omega Y)^{(n)}$. Let

$$t_n\colon \Sigma(\Omega Y)^{(n)} \longrightarrow [\Sigma\Omega Y]_n$$

be the map to the telescope. In the next proposition, we give an explicit homotopy equivalence for $\Sigma\Omega Y$. Define $\bar{H}^Y$ as the composite

$$\bar{H}^Y\colon \Sigma\Omega Y \xrightarrow{\Sigma\Omega s} \Sigma\Omega\Sigma\Omega Y \xrightarrow{\bar{H}} \bigvee_{n=1}^{\infty} \Sigma(\Omega Y)^{(n)} \xrightarrow{\bigvee_{n=1}^{\infty} t_n} \bigvee_{n=1}^{\infty} [\Sigma\Omega Y]_n.$$

Proposition 5.1. *For any $Y \in$ **CoH**, the map $\Sigma\Omega Y \xrightarrow{\bar{H}^Y} \bigvee_{n=1}^{\infty}[\Sigma\Omega Y]_n$ is a homotopy equivalence.*

Proof. Let $H_*(\Omega Y)$ be filtered by the products of the augmentation ideal. Let $H_*(\Omega\Sigma\Omega Y) = T(\widetilde{H}_*(\Omega Y))$ be filtered by

$$I^t H_*(\Omega\Sigma\Omega Y) = \sum_{t_1 r_1 + \cdots t_s r_s \geq t} (I^{t_1} H_*(\Omega Y))^{\otimes r_1} \otimes \cdots \otimes (I^{t_s} H_*(\Omega Y))^{\otimes r_s}$$

and let $H_*(\bigvee_{n=1}^{\infty}(\Omega Y)^n))$ be filtered by

$$I^t H_*(\bigvee_{n=1}^{\infty} (\Omega Y)^{(n)}) = H(\bigvee_{n=t}^{\infty} (\Omega Y)^{(n)}).$$

By [10, Proposition 3.7], the map

$$\bar{H}_*\colon H_*(\Sigma\Omega\Sigma\Omega Y) \longrightarrow H_*(\bigvee_{n=1}^{\infty} \Sigma(\Omega Y)^{(n)})$$

preserves the filtration. Since $\Omega\sigma_*$ and Ωs_* are algebra maps they also preserve the filtration and so each $(g_n)_*$ is filtration-preserving. Now, arguing as as in the proof of Lemma 3.1,

$$E^0\bar{H}^Y_*\colon E^0\widetilde{H}_*(\Sigma\Omega Y) \longrightarrow E^0\widetilde{H}_*(\bigvee_{n=1}^{\infty} [\Sigma\Omega Y]_n)$$

is an isomorphism and the result follows. □

Let

$$H^Y\colon \Omega Y \longrightarrow \Omega(\bigvee_{n=1}^{\infty} [\Sigma\Omega Y]_n))$$

be the adjoint of $\bar{H}^Y$. In Lemma 5.3 we will show that when $Y = \Sigma X$, $H^{\Sigma X}$ coincides with the James-Hopf invariant H. Before doing this we need a preliminary lemma which points out key properties of the self-map g_n. Let $j\colon X \longrightarrow \Omega\Sigma X$ be the suspension, and note that the standard co-H structure on ΣX corresponds to Σj being a right homotopy inverse of the evaluation map σ.

Lemma 5.2. *The following hold:*

(1) *for any path-connected space X, there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Sigma X^{(n)} & = & \Sigma X^{(n)} \\ \downarrow{\scriptstyle \Sigma j^{(n)}} & & \downarrow{\scriptstyle \Sigma j^{(n)}} \\ \Sigma(\Omega\Sigma X)^{(n)} & \xrightarrow{g_n} & \Sigma(\Omega\Sigma X)^{(n)} \end{array}$$

and the composite

$$\Sigma X^{(n)} \xrightarrow{\Sigma j^{(n)}} \Sigma(\Omega\Sigma X)^{(n)} \xrightarrow{t_n} [\Sigma\Omega\Sigma X]_n$$

is a homotopy equivalence;

(2) *for any* $Y \in \mathbf{CoH}$*, with* $V = \Sigma^{-1}\widetilde{H}_*(Y)$ *and* i *the inclusion* $V \longrightarrow T(V)$*, there is a commutative diagram*

$$\begin{array}{ccc} \Sigma V^{\otimes n} & = & \Sigma V^{\otimes n} \\ \downarrow{\scriptstyle \Sigma i^{\otimes n}} & & \downarrow{\scriptstyle \Sigma i^{\otimes n}} \\ E^0H_*(\Sigma(\Omega\Sigma X)^{(n)}) & \xrightarrow{E^0(g_n)_*} & E^0H_*(\Sigma(\Omega\Sigma X)^{(n)}) \end{array}$$

and the composite

$$\Sigma V^{\otimes n} \xrightarrow{\Sigma i^{\otimes n}} E^0H_*(\Sigma(\Omega\Sigma X)^{(n)}) \xrightarrow{E^0(t_n)_*} E^0H_*([\Sigma\Omega\Sigma X]_n)$$

is an isomorphism.

Proof. First, let f be the composite

$$f\colon \Sigma\Omega\Sigma\Omega Y \xrightarrow{\Sigma\Omega\sigma} \Sigma\Omega Y \xrightarrow{\Sigma\Omega s} \Sigma\Omega\Sigma\Omega Y.$$

Observe that for any $Y \in \mathbf{CoH}$, the fact that $Y \xrightarrow{s} \Sigma\Omega Y$ is a right homotopy inverse of $\Sigma\Omega Y \xrightarrow{\sigma} Y$ implies that $f \circ \Sigma\Omega s \simeq \Sigma\Omega s$.

For part (1), we have $Y = \Sigma X$ and $s = \Sigma j$. Consider the diagram

$$\begin{array}{ccccccc} \bigvee_{n=1}^{\infty} \Sigma X^{(n)} & \xrightarrow{\bar{H}^{-1}} & \Sigma\Omega\Sigma X & = & \Sigma\Omega\Sigma X & \xrightarrow{\bar{H}} & \bigvee_{n=1}^{\infty} \Sigma X^{(n)} \\ \downarrow{\scriptstyle \bigvee_{n=1}^{\infty} \Sigma j^{(n)}} & & \downarrow{\scriptstyle \Sigma\Omega\Sigma j} & & \downarrow{\scriptstyle \Sigma\Omega\Sigma j} & & \downarrow{\scriptstyle \bigvee_{n=1}^{\infty} \Sigma j^{(n)}} \\ \bigvee_{n=1}^{\infty} \Sigma(\Omega\Sigma X)^{(n)} & \xrightarrow{\bar{H}^{-1}} & \Sigma\Omega\Sigma\Omega\Sigma X & \xrightarrow{f} & \Sigma\Omega\Sigma\Omega\Sigma X & \xrightarrow{\bar{H}} & \bigvee_{n=1}^{\infty} \Sigma(\Omega\Sigma X)^{(n)} \end{array}$$

The left and right squares homotopy commute by the naturality of $\bar{H}$ while the middle square homotopy commutes by the first paragraph. Notice that the top row is homotopic to the identity map. The diagram asserting that $g_n \circ \Sigma j^{(n)} \simeq \Sigma j^{(n)}$ now follows by including the n^{th}-wedge summand into both terms on the left and pinching onto the n^{th}-wedge summand of both terms on the right.

To say that $g_n \circ \Sigma j^{(n)} \simeq \Sigma j^{(n)}$ means that $\Sigma j^{(n)}$ is invariant when composed with g^n. Now take the telescope of g_n and consider the composite $\Sigma X^{(n)} \xrightarrow{\Sigma j^{(n)}} \Sigma(\Omega\Sigma X)^{(n)} \xrightarrow{t_n} [\Sigma\Omega\Sigma X]_n$. Since $(\Sigma j^{(n)})_*$ is an injection, the invariance property of $\Sigma j^{(n)}$ for g_n implies that the composite $(t_n)_* \circ (\Sigma j^{(n)})_*$ is an injection. On the other hand, observe that for each n, the image of $(g_n)_*$ is $\Sigma\widetilde{H}_*(X^{(n)})$. Thus $(t_n)_* \circ (\Sigma j^{(n)})_*$ is an injection from $\Sigma\widetilde{H}_*(X^{(n)})$ onto itself, and so it is an isomorphism. Thus $t_n \circ \Sigma j^{(n)}$ is a homotopy equivalence.

Part (2) is similar, using the facts from Section 2 that, on the level of the associated graded, s_* is the suspension of the inclusion $V \xrightarrow{i} T(V)$ and σ_* is the suspension of the projection $T(V) \longrightarrow V$. □

The homotopy equivalence $l = t_n \circ \Sigma j^{(n)}$ in Lemma 5.2 (1) lets us equivalently replace $[\Sigma\Omega\Sigma X]_n$ by $\Sigma X^{(n)}$ and t_n by $t'_n = t_n \circ l^{-1}$.

Lemma 5.3. *Let* X *be a path-connected space. Then* $H^{\Sigma X} \simeq H$*.*

Proof. It is equivalent to adjoint and prove that $\bar{H}^{\Sigma X} \simeq \bar{H}$. Consider the diagram

$$\begin{array}{ccccc} \Sigma\Omega\Sigma X & \xrightarrow{\bar{H}} & \bigvee_{n=1}^{\infty} \Sigma X^{(n)} & =\!=\!= & \bigvee_{n=1}^{\infty} \Sigma X^{(n)} \\ \downarrow{\scriptstyle \Sigma\Omega\Sigma j} & & \downarrow{\scriptstyle \bigvee_{n=1}^{\infty} \Sigma(\Omega\Sigma j)^{(n)}} & & \| \\ \Sigma\Omega\Sigma\Omega\Sigma X & \xrightarrow{\bar{H}} & \bigvee_{n=1}^{\infty} \Sigma(\Omega\Sigma X)^{(n)} & \xrightarrow{\bigvee_{n=1}^{\infty} t'_n \circ g_n} & \bigvee_{n=1}^{\infty} [\Sigma\Omega\Sigma X]_n. \end{array}$$

The left square homotopy commutes by the naturality of $\bar{H}$ and the right square homotopy commutes by Lemma 5.2 (1) and the definition of t'_n. The lower direction around the diagram is the definition of $\bar{H}^{\Sigma X}$ while the upper direction around the diagram is simply $\bar{H}$, so $\bar{H}^{\Sigma X} \simeq \bar{H}$. □

For $Y \in \mathbf{CoH}$, define the n-th Hopf invariant

$$H_n^Y : \Omega Y \longrightarrow \Omega[\Sigma\Omega Y]_n$$

as the adjoint of the composite

$$\Sigma\Omega Y \xrightarrow{\bar{H}^Y} \bigvee_{n=1}^{\infty} \mathrm{hocolim}_{g_n} \Sigma(\Omega Y)^{(n)} \xrightarrow{\pi} \mathrm{hocolim}_{g_n} \Sigma(\Omega Y)^{(n)} = [\Sigma\Omega Y]_n$$

where π is the pinch map. We wish to determine the behavior of H_n^Y in homology. When $Y = \Sigma X$, Lemma 5.3 implies that $H_n^{\Sigma Y} \simeq H_n$, and $(H_n)_*$ was described in [10] in terms of an algebraic James-Hopf map. In Theorem 1.5 we will show this description generalizes for any $Y \in \mathbf{CoH}$. To start, we begin by recalling some material from [10].

Let $\mathbf{k}$ be the ground field. A coalgebra will refer to a graded cocommutative coalgebra and a filtration of a module M will refer to a decreasing filtration with $I^0M = M$. A *pointed filtered coalgebra* D is a filtered module D with a filtered comultiplication $\psi\colon D \longrightarrow D \otimes D$ turning D into an augmented coalgebra with a filtered unit and a filtered augmentation. Given a pointed filtered coalgebra D, the *algebraic James construction* $J(D)$ is defined as the coequalizer of the diagram with morphisms

$$s_i^{n-1}\colon C^{\otimes n-1} = C \otimes \cdots \otimes C \otimes \overset{(i)}{\mathbf{k}} \otimes C \otimes \cdots \otimes C \longrightarrow C^{\otimes n}$$

for $1 \le i \le n < \infty$. Observe that $J(D)$ is the coadjoint of the forgetful functor from filtered Hopf algebras to pointed filtered coalgebras, which has the universal property that for any pointed filtered coalgebra map $f\colon D \to B$ with B a Hopf algebra there is a unique Hopf algebra map $Jf\colon J(D) \longrightarrow B$ such that $Jf|_D = f$. For pointed filtered coalgebras D and D', let

$$D \vee D' = D \otimes \mathbf{k} + \mathbf{k} \otimes D' \subseteq D \otimes D'$$

be the coproduct of pointed coalgebras. Define the smash product $D \wedge D'$ to be the coalgebra cokernel of the inclusion $D \vee D' \longrightarrow D \otimes D'$. The algebraic James-Hopf map

$$\mathcal{H}\colon J(D) \longrightarrow J(\bigvee_{n=1}^{\infty} D^{\wedge n})$$

is defined by exactly mimicking James' definition of the combinatorial James-Hopf invariant. Let $\mathcal{H}_n$ be the composite

$$\mathcal{H}_n\colon J(D) \xrightarrow{\mathcal{H}} J(\bigvee_{n=1}^{\infty} D^{\wedge n}) \longrightarrow J(D^{\wedge n}).$$

The key property of $\mathcal{H}$ is the following, proved in [10, Prop 3.7].

Lemma 5.4. *The algebraic James-Hopf map coincides with the geometric James-Hopf map in homology. That is, there is an equality of maps*

$$H_*, \mathcal{H}\colon H_*(\Omega\Sigma X) = J(D) \longrightarrow H_*(\Omega\Sigma(\bigvee_{n=1}^{\infty} X^{(n)})) = J(\bigvee_{n=1}^{\infty} D^{\wedge n}).$$

□

There is a filtration on $J(D)$ given by tensor length, $J_0(D) = \mathbf{k} \subseteq J_1(D) = D \subseteq J_2(D) \subseteq \cdots \subseteq J(D)$. As well, each $J_n(D)$ has a filtration induced by the one on D, and the inclusion $J_n(D) \longrightarrow J_{n+1}(D)$ is a morphism of pointed filtered coalgebras. In [10, Props 3.6,3.8] it is shown that the two filtrations are compatible, in the sense that there are equalities

(1) $E^0 J(D) = J(E^0 D)$;
(2) $E^0\mathcal{H} = \mathcal{H}\colon E^0 J(D) = J(E^0 D) \longrightarrow E^0 J(\bigvee_{n=1}^{\infty} D^{\wedge n}) = J(\bigvee_{n=1}^{\infty} (E^0 D)^{\wedge n})$.

In practise, if X is a path-connected space, we let $D = \widetilde{H}_*(X)$ and filter D by setting $I^0 D = \mathbf{k} \oplus \widetilde{H}_*(X)$, $I^1 D = \widetilde{H}_*(X)$, and $I^t D = 0$ for $t > 1$, and apply (1) and (2) in the context of $H_*(\Omega\Sigma X) = J(D)$.

Now consider the case when $Y \in$ **CoH**. Let $A = H_*(\Omega Y)$ as a Hopf algebra and let $C = \mathbf{k} \oplus \Sigma^{-1}\widetilde{H}_*(Y)$ as a graded module with the trivial comultiplication, where the ground field is $\mathbf{k} = \mathbb{Z}/p$ and $C_0 = \mathbf{k}$. Let A be filtered by the products of the augmentation ideal and let C be filtered by $I^0 C = C$, $I^1 C = \Sigma^{-1}\widetilde{H}_*(Y)$, and $I^t C = 0$ for $t > 1$. Observe that

$$H_*(\Omega\Sigma\Omega Y) = J(A)$$

because $H_*(\Omega\Sigma\Omega Y)$ satisfies the universal property for the functor J on A. Before describing $(H_n^Y)_*$ in Theorem 1.5 we need a preliminary lemma.

Since $E^0 A$ is primitively generated by $\Sigma^{-1}\widetilde{H}_*(Y)$, we have

$$E^0 A = T(\Sigma^{-1}\widetilde{H}_*(Y)) = J(C).$$

Let ϕ be the map

$$\phi = (\Omega s)_*\colon H_*(\Omega Y) = A \longrightarrow H_*(\Omega\Sigma\Omega Y) = J(A).$$

Then $E^0\phi\colon E^0 A \longrightarrow E^0 J(A) = J(E^0 A)$ is a morphism of bigraded Hopf algebras which induces a bigraded map

$$QE^0\phi\colon QE^0 A \longrightarrow QE^0 J(A)$$

of modules of indecomposable elements. Since the composite

$$IE^0 A \hookrightarrow IJ(E^0 A) = IE^0 J(A) \twoheadrightarrow Q(E^0 J(A))$$

is an isomorphism of bigraded modules, we have

$$QE_1^0 J(A) = IE_1^0 A = IC.$$

Thus $E^0\phi\colon E^0A \longrightarrow E^0J(A)$ is the unique map of Hopf algebras induced by the composite of inclusions

$$C \xrightarrow{j} E^0A \longrightarrow J(E^0A) = J(J(C)).$$

Hence the uniqueness property of the functor J implies the following.

Lemma 5.5. *The two maps $E^0\phi, J(j)\colon J(C) \longrightarrow J(J(C))$ are equal.* □

Proof of Theorem 1.5: Observe that H_n^Y is homotopic to the composite

$$\Omega Y \xrightarrow{\Omega s} \Omega\Sigma\Omega Y \xrightarrow{H} \Omega\Sigma\left(\bigvee_{n=1}^{\infty}(\Omega Y^{(n)})\right) \xrightarrow{\Omega\pi} \Omega\Sigma(\Omega Y)^{(n)} \xrightarrow{\Omega g_n} \Omega\Sigma(\Omega Y)^{(n)} \xrightarrow{\Omega t_n} \Omega[\Sigma\Omega Y]_n$$

where π is the pinch map. We examine the induced map in homology. As before, let $C = \mathbf{k}\oplus\Sigma^{-1}\widetilde{H}_*(Y)$ and $A = H_*(\Omega Y)$. By Lemma 5.4, the algebraic James-Hopf map coincides with the geometric James-Hopf map in homology and so

$$H_* = \mathcal{H}\colon H_*(\Omega\Sigma\Omega Y) = J(A) \longrightarrow \Omega\Sigma\left(\bigvee_{n=1}^{\infty}(\Omega Y)^{(n)}\right) = J\left(\bigvee_{n=1}^{\infty}A^{\wedge n}\right).$$

Consider the diagram

$$\begin{array}{ccccc}
J(C) & & \xrightarrow{\quad\mathcal{H}\quad} & & J\left(\bigvee_{n=1}^{\infty}(C)^{\wedge n}\right)\\
\| & \searrow^{J(j)} & & & \downarrow{\scriptstyle J(\vee_{n=1}^{\infty}j^{\wedge n})}\\
E^0A & \xrightarrow{E^0\phi} & J(J(C)) & \xrightarrow{E^0\mathcal{H}=\mathcal{H}} & J\left(\bigvee_{n=1}^{\infty}(J(C))^{\wedge n}\right)\\
\| & & & & \|\\
E^0H_*(\Omega Y) & & \xrightarrow{E^0(H_*\circ(\Omega s)_*)} & & E^0H_*\left(\Omega\Sigma\left(\bigvee_{n=1}^{\infty}(\Omega Y)^{(n)}\right)\right).
\end{array}\tag{5.1}$$

The upper left triangle commutes by Lemma 5.5, the upper right quadrilateral commutes by the naturality of $\mathcal{H}$, and the lower rectangle commutes because by definition $\phi = (\Omega s)_*$ while $H_* = \mathcal{H}$. Now project the right column in (5.1) to the n^{th}-term and consider the diagram

(5.2)

$$\begin{array}{ccccc}
J((C)^{\wedge n}) & = & J((C)^{\wedge n}) & & \\
\downarrow{\scriptstyle J(j^{\wedge n})} & & \downarrow{\scriptstyle J(j^{\wedge n})} & \searrow^{\cong} & \\
J((J(C))^{\wedge n}) & & J((J(C))^{\wedge n}) & & \\
\| & & \| & & \\
E^0H_*(\Omega\Sigma(\Omega Y)^{(n)}) & \xrightarrow{E^0(\Omega g_n)_*} & E^0H_*(\Omega\Sigma(\Omega Y)^{(n)}) & \xrightarrow{E^0(\Omega t)_*} & E^0H_*(\Omega[\Sigma\Omega Y]_n).
\end{array}$$

As all maps are algebra maps, the diagram will commute provided it does when restricted to $C^{\wedge n}$. Commutativity now follows from Lemma 5.2 (2). Combining (5.1), its projection onto the n^{th}-term, and (5.2) shows that there is a commutative diagram

$$\begin{array}{ccc} J(C) & \xrightarrow{\mathcal{H}_n} & J((C)^{\otimes n}) \\ \downarrow{\cong} & & \downarrow{\cong} \\ E^0H_*(\Omega Y) & \xrightarrow{E^0H^Y_{n*}} & E^0H_*(\Omega[\Sigma\Omega Y]_n). \end{array}$$

Since the top row is identical to $T(\Sigma^{-1}\widetilde{H}_*(Y)) \xrightarrow{\mathcal{H}_n} T((\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n})$, the theorem is proved. □

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