Artin braid groups and the homotopy groups –Joint with Jingyan Li

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Boundary Brunnian braids, mirror reflection and the homotopy groups

Combinatorial Descriptions of Homotopy Groups

- **Group** $G(n)$ is combinatorially defined by:

  Generators: $x_1, x_2, \ldots, x_n$;
  Relations: (1) the ordered product of the generators $x_1 x_2 \cdots x_n = 1$ and (2) the **iterated commutators** on the generators

  $$[x_{i_1}^{\epsilon_1}, x_{i_2}^{\epsilon_2}, \ldots, x_{i_t}^{\epsilon_t}] = 1,$$

  where **each generator occurs at least once** in the commutator bracket.

- **Theorem (-).** $\pi_n(S^2)$ is isomorphic to the center of $G(n)$ for each n.
- **Remark.** The result was first given in my thesis (University of Rochester 1995) published in Math. Proc. Camb. Phil. Soc. 130 (2001), no.3, 489-513.
- **Question.** It was then asked by many people whether there is a **finitely presented** group whose center is given by $\pi_n(S^2)$. We will give a **positive answer**.

Homotopy Groups and the Fixed-point Set

- The **Artin Representation** of the braid group B_n on the free group F_n generated by $x_1, x_2, \ldots, x_n$ induces a B_n-action on the group $G(n)$.
- **Theorem (-).** Let $n \geq 4$. Then
  1) the center of $G(n)$ is the fixed set of the pure braid group action on $G(n)$ and so is $\pi_n(S^2)$;
  2) the fixed-point set of the braid group action on $G(n)$ is the subgroup $\{x \in Z(G(n)) | 2x = 0\}$ of the center $Z(G(n))$.
- **Remark.** The braid group action on $G(n)$ was observed by myself in 1997. Then Larry Taylor conjectured to Fred Cohen that the center is given by the fixed-point set of the braid group action.
- **Remark.** The result was published in Proc. London Math. Soc., (3) 84 (2002), no. 3, 645–662.

Remarks on the Homotopy Groups

- It was known by Serre that $\pi_n(S^2)$ is a finite abelian group for $n \geq 4$.
- Let $\mathrm{Tor}_p(G)$ denote the p-torsion component of an abelian group G for a prime integer p.
- It was proved by James that $4 \cdot \mathrm{Tor}_2(\pi_n(S^2)) = 0$. Namely $\mathrm{Tor}_2(\pi_n(S^2))$ is a finite direct sum of $\mathbb{Z}/2$ and $\mathbb{Z}/4$.
- Furthermore, a result of Selick states that $p \cdot \mathrm{Tor}_p(\pi_n(S^2)) = 0$ for the prime $p > 2$, namely $\mathrm{Tor}_p(\pi_n(S^2))$ is a finite direct sum of $\mathbb{Z}/p$ for $p > 2$.
- Thus, for $n \geq 4$, $\pi_n(S^2)$ is determined by the **orders of the groups** $|\pi_n(S^2)|$ and $|\{\alpha \in \pi_n(S^2) \mid 2\alpha = 0\}|$.
- The subgroup $\{\alpha \in \pi_n(S^2) \mid 2\alpha = 0\}$ has some special geometric meanings. For instance, the non-suspension co-H-spaces of 3-cell complexes with bottom two-cell given by the suspension of $\mathbb{R}\mathrm{P}^2$ are classified by these groups.

Braid Groups and Pure Braid Groups

- The **braid group** B_n is generated by

$$\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$$

  subject to the braiding relations:
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$, and
  - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for each i.
- The **symmetric group** S_n is the quotient group of B_n subject to the following additional relations:
  - $\sigma_i^2 = 1$ for each i.
- The **pure braid group** P_n is defined to be the kernel of the quotient map $B_n \to S_n$, with a set of generators given by

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

  for $1 \leq i < j \leq n$.

Brunnian Braids

- A (geometric) braid is called **Brunnian** if (1) it is a pure braid, and (2) it becomes a trivial braid by removing any of its strands. Since the composition of any two Brunnian braids is still Brunnian, the set of Brunnian braids is a (normal) subgroup of B_n which is denoted by Brun_n.
- **Theorem (Berrick-Cohen-Wong-Wu).** Let $n \geq 4$. Then there is an exact sequence of groups

$$1 \to \mathrm{Brun}_{n+2}(S^2) \to \mathrm{Brun}_{n+1} \to \mathrm{Brun}_{n+1}(S^2) \to \pi_n(S^2) \to 1,$$

  where $\mathrm{Brun}_n(S^2)$ is the group of Brunnian braids over the 2-sphere.
- **Remark.** The result was published in J. Amer. Math. Soc. 19 (2006), no. 2, 265–326.
- **Remark.** Fred Cohen and I were starting to work Brunnian braids and homotopy in 2000, while Fred's wife brought some threads for us to make braids.

Normal Genrators of Brunnian Braids

- **Theorem.** The Brunnian group Brun_n is the **normal subgroup** of the pure braid group P_n generated by the iterated commutators

$$[[[A_{1,2}, A_{i_2,3}], A_{i_3,4}], \ldots, A_{i_{n-1},n}]$$

  for $1 \leq i_t \leq t$ and $2 \leq t \leq n-1$.
- Thus the quotient groups P_n/Brun_n and B_n/Brun_n are **finitely presented.**
- **Remark.** As a group, Brun_n is a free group of rank ∞ for $n \geq 3$.
- **Remark.** The above theorem is one of the results in our new preprint.

Boundary Brunnian braids

- Let $\Gamma_{0,n}$ be the mapping class group of n-punctured sphere. Consider B_n as the mapping class group of n-punctured disk. The canonical embedding of the disk into the sphere (as northern hemisphere) induces a group homomorphism $q\colon B_n \to \Gamma_{0,n}$.
- **Boundary Brunnian Braids:** Let $\mathrm{Bd}_n = [\mathrm{Brun}_n, \mathrm{Ker}(q)]$ be the commutator subgroup in B_n of Brun_n and the kernel of q.
- **Proposition:** Bd_n is the normal subgroup of P_n generated by the iterated commutators

$$[[[A_{1,2}, A_{i_2,3}], A_{i_3,4}, \ldots, A_{i_{n-1},n}], A_{0,i_n}]$$

for $1 \leq i_t \leq t$ and $2 \leq t \leq n$.
- Thus the groups P_n/Bd_n and B_n/Bd_n are **finitely presented.**

Center Theorem

For a subgroup H of G, let

$$\sqrt{(H, G)} = \{x \in G \mid x^q \in H \text{ for some } q \in \mathbb{Z}\}$$

denote the set of the roots of H in G. Denote by $Z(G)$ the center of a group G.

- **Theorem.** Let $n \geq 4$. $\sqrt{(\mathrm{Bd}_n, P_n)} / \mathrm{Bd}_n \cong \pi_n(S^2)$.
- **Theorem.** Let $n \geq 4$. There are isomorphisms of groups

$$Z(P_n / \mathrm{Bd}_n) \cong \pi_n(S^2) \times \mathbb{Z},$$

$$Z(B_n / \mathrm{Bd}_n) \cong \{\alpha \in \pi_n(S^2) \mid 2\alpha = 0\} \times \mathbb{Z}.$$

The method of the proof uses

- simplicial techniques and
- computations on the conjugations of braids.

Mirror Reflection Problem

- By moving our steps to the next, consider the mirror reflection $\chi\colon B_n \to B_n$. Given a subgroup G of B_n, one may ask what are the mirror symmetric braids β subject to G, that is, the braids β satisfying the equation of cosets $\chi(\beta)G = \beta G$.
- If G is the trivial subgroup, it is well-known that the mirror reflection χ on B_n is free and so the trivial braid is the only mirror symmetric braid subject to the trivial subgroup $G = \{1\}$.
- For general cases that G is a non-trivial subgroup of B_n, the question on mirror symmetric braids becomes very nontrivial.
- In the case $G = \mathrm{Bd}_n$, the answer is again given in term of the homotopy group $\pi_n(S^2)$.

Mirror Symmetry Theorem

- Let $\mathrm{Fix}^{\phi}(G)$ denote the subgroup of the fixed-points of an action ϕ on a group G.
- **Theorem.** The subgroup Bd_n is invariant under the mirror reflection χ. Moreover there is an isomorphism of groups

$$\mathrm{Fix}^{\chi}(B_n/\mathrm{Bd}_n) \cong \pi_n(S^2).$$

  for $n \geq 3$.
- **Remark.** The method of the proof is a combination of
  - homotopy theory on Hopf invariants,
  - simplicial techniques, and
  - computations on braids.

- **End. Thank You.**