EHP Sequences for \((p - 1)\)-cell Complexes

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EHP-fibrations

- For spaces localized at 2, the classical EHP fibrations
  \[ \Omega^2 S^{2n+1} \xrightarrow{P} S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1} \]
  play a crucial role for the computations of the homotopy groups of the spheres.

- We will give the EHP-fibrations for \((p - 1)\)-cell complexes for \(p > 2\), which can be regarded as the odd prime analogue of the classical EHP-fibrations by considering the spheres as 1-cell complexes for \(p = 2\).

- The new EHP-fibrations are one of the consequences of our study on the representation theory in homotopy.

- These fibrations will be obtained from the evaluations of the functor \(A^{\min}\) on \((p - 1)\)-cell complexes.
The Algebraic Functor $A^{\min}$

- The algebraic functor $A^{\min}$ was introduced by Paul Selick and myself, arising from the question on the naturality of the classical Poincaré-Birkhoff-Witt isomorphism.
- For any ungraded module $V$, $A^{\min}(V)$ is defined to be the smallest functorial coalgebra retract of $T(V)$ containing $V$.
- Then the functor $A^{\min}$ extends canonically to the cases when $V$ is any graded module.
- The functor $A^{\min}$ admits the tensor-length decomposition with

$$A^{\min}(V) = \bigoplus_{n=0}^{\infty} A^{\min}_n(V),$$

where $A^{\min}_n(V) = A^{\min}(V) \cap T_n(V)$ is the homogenous component of $A^{\min}(V)$. 

The Algebraic Problem

• By the Functorial Poincaré-Birkhoff-Witt Theorem, there exists a functor $B^{\text{max}}$ from (graded) modules to Hopf algebras with the functorial coalgebra decomposition

$$T(V) \cong A^{\min}(V) \otimes B^{\text{max}}(V)$$

for any graded module $V$.

• Problem. The determination of $A^{\min}(V)$ for general $V$ is equivalent to an open problem in the modular representation theory of the symmetric groups according to, which seems beyond the reach of current techniques.

• However we are able to determine $A^{\min}(V)$ in the special cases when $V_{\text{even}} = 0$ and $\dim V = p - 1$. 
Denote by $L(V)$ the free Lie algebra generated by $V$. Write $L_n(V)$ for the $n$-th homogeneous component of $L(V)$. Observe that $[L_s(V), L_t(V)]$ is a submodule of $L_{s+t}(V)$ under the Lie bracket of $L(V)$. Let

$$\bar{L}_n(V) = L_n(V)/\sum_{i=2}^{n-2}[L_i(V), L_{n-i}(V)].$$

Define $\bar{L}_n^k(V)$ recursively by $\bar{L}_n^1(V) = \bar{L}_n(V)$ and $\bar{L}_n^{k+1}(V) = \bar{L}_n(\bar{L}_n^k(V))$. 
Theorem

Let the ground field $k$ be of characteristic $p > 2$. Let $V$ be a graded module such that $V_{\text{even}} = 0$ and $\dim V = p - 1$. Then there is an isomorphism of coalgebras

$$A^{\min}(V) \cong \bigotimes_{k=0}^{\infty} E(\bar{L}^k_p(V)),$$

where $E(W)$ is the exterior algebra generated by $W$. 
Geometric Realization Theorem

There exist homotopy functors $\bar{A}^{\text{min}}$ and $\bar{B}^{\text{max}}$ from simply connected co-$H$ spaces of finite type to spaces such that for any $p$-local simply connected co-$H$ space $Y$ the following hold:

1) $\bar{A}^{\text{min}}(Y)$ is a functorial retract of $\Omega Y$ and so there is a functorial decomposition

$$\Omega Y \cong \bar{A}^{\text{min}}(Y) \times \bar{B}^{\text{max}}(Y).$$

2) On mod $p$ homology the decomposition

$$H_*(\Omega Y) \cong H_*(\bar{A}^{\text{min}}(Y)) \otimes H_*(\bar{B}^{\text{max}}(Y))$$

is with respect to the augmentation ideal filtration.

3) On mod $p$ homology the associated bigraded $E^0 H_*(\bar{A}^{\text{min}}(Y))$ is given by

$$E^0 H_*(\bar{A}^{\text{min}}(Y)) = A^{\text{min}}(\Sigma^{-1} \bar{H}_*(Y)).$$
Corollary

Let $X$ be a path-connected finite complex. Define

$$b_X = \sum_{q=1}^{\infty} q \dim \tilde{H}_q(X; \mathbb{Z}/p).$$

Roughly speaking, $b_X$ is the summation of the dimensions of the cells in $X$.

**Theorem.** Let $p > 2$ and let $Y$ be any $p$-local simply connected co-$H$ space. Suppose that $\tilde{H}_{\text{odd}}(Y) = 0$ and $\dim \tilde{H}_*(Y) = p - 1$. Then there is an isomorphism of coalgebras

$$H_*(\bar{A}^{\text{min}}(Y)) \cong E(\Sigma^{-1} \tilde{H}_*(Y)) \otimes \bigotimes_{k=1}^{\infty} E(\Sigma^{\frac{p^k-1}{p-1}} b_Y - p^k \tilde{H}_*(Y)),$$

where $\Sigma^{\frac{p^k-1}{p-1}} b_Y - p^k \tilde{H}_*(Y) = \bar{L}_p^k(\Sigma^{-1} \tilde{H}_*(Y))$. 
Let $Y$ be any $p$-local simply connected co-$H$ space. Then there is a suspension splitting

$$\Sigma \tilde{A}_{\text{min}}(Y) \simeq \bigvee_{n=1}^{\infty} \tilde{A}_n^{\text{min}}(Y)$$

such that

$$\Sigma^{-1} \tilde{H}_*(\tilde{A}_n^{\text{min}}(Y)) \simeq A_n^{\text{min}}(\Sigma^{-1} \tilde{H}_*(Y))$$

for each $n \geq 1$. 
EHP Fibration

Let \( p > 2 \) and let \( Y \) be any \( p \)-local simply connected co-\( H \) space. Suppose that \( \tilde{H}_{\text{odd}}(Y) = 0 \) and \( \dim \tilde{H}_{\ast}(Y) = p - 1 \). Then there is a fibre sequence

\[
\Omega \tilde{A}_{\text{min}}^{\text{min}}(\Sigma^{b_{Y}-p+1}Y) \xrightarrow{P} \tilde{E}(Y) \xrightarrow{E} \tilde{A}_{\text{min}}^{\text{min}}(Y) \xrightarrow{H_{p}} \tilde{A}_{\text{min}}^{\text{min}}(\Sigma^{b_{Y}-p+1}Y).
\]

with the following properties:

1) On mod \( p \) homology \( H_{\ast}(\tilde{E}(Y)) \cong E(\Sigma^{-1} \tilde{H}_{\ast}(Y)) \) as coalgebras.

2) \( H_{\ast}(\tilde{A}_{\text{min}}^{\text{min}}(Y)) \cong H_{\ast}(\tilde{E}(Y)) \otimes H_{\ast}(\tilde{A}_{\text{min}}^{\text{min}}(\Sigma^{b_{Y}-p+1}Y)) \) as coalgebras.

3) If \( f: S^{n} \to Y \) is a co-\( H \) map such that \( f_{\ast} \neq 0: \tilde{H}_{\ast}(S^{n}) \to \tilde{H}_{\ast}(Y) \).
• Then there is a commutative diagram of fibre sequences

\[
\begin{array}{cccc}
\Omega \bar{A}^{\min}(\Sigma^{b_Y-p+1}Y) & \xrightarrow{P} & \bar{E}(Y) & \xrightarrow{E} \bar{A}^{\min}(Y) & \xrightarrow{H_p} \bar{A}^{\min}(\Sigma^{b_Y-p+1}Y) \\
| & | & | & | & |
\Omega \bar{A}^{\min}(\Sigma^{b_Y-p+1}Y) & \xrightarrow{P_f} S^{n-1} & B_f & \xrightarrow{} \bar{A}^{\min}(\Sigma^{b_Y-p+1}Y).
\end{array}
\]

In particular, the map \( P : \Omega \bar{A}^{\min}(\Sigma^{b_Y-p+1}Y) \to \bar{E}(Y) \) factors through the bottom cell of \( \bar{E}(Y) \).

• THANK YOU