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COHEN GROUPS

JIE WU

Department of Mathematics
National University of Singapore
Singapore 117543
Republic of Singapore

matwuj@nus.edu.sg

<http://www.math.nus.edu.sg/~matwujie>

This talk consists of:

1. The Cohen Groups.
2. Algebraic Representations of Cohen Groups.

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1. The Cohen Groups

The Cohen groups arises from attacking the following exponent conjectures:

Barratt Conjecture. Let $f: \Sigma^2 X \rightarrow Y$ be a map such that $[f]$ is of order p^r in $[\Sigma^2 X, Y]$, that is $p^r \cdot [f] = 0$. Then $p^{r+1} \text{Im}(f_*: \pi_*(\Sigma^2 X) \rightarrow \pi_*(Y)) = 0$.

James Construction. For any pointed space X , the James construction $J(X)$ is the free monoid generated by X modulo the single relation $* = 1$, with weak topology. If X is path-connected, then $J(X) \simeq \Omega \Sigma X$. **James filtration is just the word-length filtration** and so $J_n(X)$ is the quotient of X^n by equivalence relations like

$$*x_1x_2 \sim x_1 * x_2 \sim x_1x_2* \sim x_1x_2.$$

From the James filtration

$$\begin{array}{c} X^n \\ \downarrow q_n \\ X = J_1(X) \subseteq \cdots \subseteq J_n(X) \subseteq \cdots \subseteq J(X), \end{array}$$

there is a tower of groups

$$\begin{array}{c}
 [J(X), \Omega Y] \twoheadrightarrow \cdots \twoheadrightarrow [J_n(X), \Omega Y] \twoheadrightarrow \cdots \twoheadrightarrow [X, \Omega Y] \\
 \downarrow q_n^* \\
 [X^n, \Omega Y]
 \end{array}$$

by suspension splitting theorem, that is,

$$[J(X), \Omega Y] = \lim_n [J_n(X), \Omega Y]$$

with representation

$$q_n^*: [J_n(X), \Omega Y] \hookrightarrow [X^n, \Omega Y].$$

The coordinate inclusions

$$d^i: X^{n-1} \rightarrow X^n \quad (x_0, x_1, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, *, x_i, \dots, x_{n-1})$$

induces faces $d_i = d^{i*}: [X^n, Z] \rightarrow [X^{n-1}, Z]$.

Lemma. Let X and Z be path-connected spaces. Suppose that X is a co- H -space or Z is an H -space. Then $[J_n(X), Z]$ is the equalizer of the faces $d_i = d^{i*}: [X^n, Z] \rightarrow [X^{n-1}, Z]$ for $0 \leq i \leq n$.

Given any map $f: \Sigma X \rightarrow Y$, he constructed a group homomorphism from the free group $\theta_f: F_n \rightarrow [X^n, \Omega Y]$ by sending the generators to

$$\theta_f(x_i): X^n \xrightarrow{i\text{-th coordinate projection}} X \hookrightarrow \Omega \Sigma X \xrightarrow{\Omega f} \Omega Y.$$

$$\text{Then } \Omega f|_{J_n(X)} = \theta(x_1 x_2 \cdots x_n).$$

Suppose that $\bar{\Delta}: X \rightarrow X \wedge X$ is null-homotopic (for instance X is a suspension). Then θ_f factors through the quotient K_n of F_n subject to the relations:

$$[[x_{i_1}, x_{i_2}], x_{i_3}], \dots, x_{i_i}] = 1$$

if any one of generators occurs at least twice.

Suppose that $p^r[f] = 0$. Then θ_f factors through the quotient $K_n^{p^r}$ of K_n subject to the second type relations: $x_i^{p^r} = 1$ for $1 \leq i \leq n$.

The **Cohen Group** \mathfrak{H}_n is obtained by taking equalizers of $d_i: K_n \rightarrow K_{n-1}$. $\mathfrak{H}_n^{p^r}$ is the equalizer of $d_i: K_n^{p^r} \rightarrow K_{n-1}^{p^r}$. Note that there is a representation

$$\theta: \mathfrak{H}_n \rightarrow [J_n(X), \Omega Y]$$

for any suspension X and any given map $f: X \rightarrow \Omega Y$.

Examples. $K_1 = \mathbb{Z}$ and $K_1^{p^r} = \mathbb{Z}/p^r$. From $[[x_1, x_2], x_1] = [[x_1, x_2], x_2] = 1$, $\Gamma_3 K_2 = 1$ and there are short exact sequences

$$\mathbb{Z}\langle [x_1, x_2] \rangle \hookrightarrow K_2 \twoheadrightarrow \mathbb{Z}\langle x_1, x_2 \rangle,$$

$$\mathbb{Z}/p^r \langle [x_1, x_2] \rangle \hookrightarrow K_2^{p^r} \twoheadrightarrow \mathbb{Z}/p^r \langle x_1, x_2 \rangle.$$

$\mathfrak{H}_2^{p^r}$ is the equalizer of $d_1, d_2: K_2^{p^r} \rightarrow K_1^{p^r} = \mathbb{Z}$ with

$$(x_1 x_2)^2 = x_1 x_2 x_1 x_2 = x_1 x_1 x_2 [x_2, x_1] x_2 = x_1^2 x_2^2 [x_2, x_1].$$

If $p > 2$, then $(x_1 x_2)^{p^r} = 1$. If $p = 2$, then $(x_1 x_2)^{2^{r+1}} = 1$. It follows that, by $\Omega f|_{J_2(X)} = \theta_f(x_1 x_2)$,

$$2^{r+1} \cdot \Omega f|_{J_2(X)} \simeq *: J_2(X) \rightarrow \Omega Y \quad \text{and}$$

$$p^r \cdot \Omega f|_{J_2(X)} \simeq *: J_2(X) \rightarrow \Omega Y.$$

In particular,

$$2^{r+1} \cdot \text{Im}(f_*: \pi_k(\Sigma X) \rightarrow \pi_k(Y)) = 0$$

for $k < \text{three times the connectivity (double stable range)}$.

(This gives a very short and easy proof for the classical theorem of Barratt!!)

★Cohen Theorem. \mathfrak{H} is a progroup and the kernel of $\mathfrak{H}_n \rightarrow \mathfrak{H}_{n-1}$ is isomorphic to $\text{Lie}(n)$, where $\text{Lie}(n)$ over a ring R (in this case $R = \mathbb{Z}$) is given as follows:

Let V be a free n -dimensional R -module with a basis $\{x_1, \dots, x_n\}$. Then $\text{Lie}(n)$ is the R -submodule of $V^{\otimes n}$ spanned by Lie elements

$$[[x_{\sigma(1)}, x_{\sigma(2)}], \dots, x_{\sigma(n)}]$$

for $\sigma \in S_n$. The symmetric group S_n acts on $\text{Lie}(n)$ by permuting letters $\{x_1, \dots, x_n\}$.

- The elements in \mathfrak{H} are given by (possibly infinite) products of composites

$$J(X) \xrightarrow{H_n} J(X^{(n)}) \xrightarrow{J(\alpha)} J(X^{(n)}) \xrightarrow{\Omega W_n} J(X) \xrightarrow{\tilde{f}} \Omega Y,$$

where H_n is the James-Hopf map, W_n is the Whitehead product, \tilde{f} is the H -map such that $\tilde{f}|_X \simeq f$ and α is a linear combination of permutations.

2. Algebraic Representation of Cohen groups

• Let ground-ring R be any commutative ring with identity and let V be a projective R -module. The tensor algebra $T(V)$ is a Hopf algebra by saying V primitive.

• Let \mathfrak{H}^R be the set of all functorial coalgebra self-maps of $T(V)$, that is, consisting of

$$\phi_V: T(V) \rightarrow T(V)$$

such that ϕ_V is a coalgebra map for each V and

$$\begin{array}{ccc} T(V) & \xrightarrow{\phi_V} & T(V) \\ \downarrow T(f) & & \downarrow T(f) \\ T(V') & \xrightarrow{\phi_{V'}} & T(V) \end{array}$$

commutes for any R -linear map $f: V \rightarrow V'$.

• \mathfrak{H}^R is a group: For any cocommutative coalgebra C and Hopf algebra D with coalgebra maps $f, g: C \rightarrow D$, **convolution product**

$$f * g: C \xrightarrow{\psi} C \otimes C \xrightarrow{f \otimes g} D \otimes D \xrightarrow{\text{multi}} D.$$

- (Selick-Wu in the AMS Memoirs paper) If $R = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{(p)}$ (p -local integers), \mathbb{Z}_p (p -complete integers) or \mathbb{Z}/p^r , then \mathfrak{H}^R is isomorphic to the Cohen group as pro-groups. (The algebraic filtration is obtained by word-length filtration from tensor algebras.)

- This tells that any functorial coalgebra decomposition of tensor algebras admits a geometric realization in the sense that: If $T(V) \cong A(V) \otimes B(V)$, then there is a functorial decomposition $\Omega\Sigma X \simeq A(X) \times B(X)$ such that $H_*(A(X)) \cong A(\bar{H}_*(X))$ and $H_*(B(X)) \cong B(\bar{H}_*(X))$.

- Moreover we solved the Cohen conjecture positively that: Let $A^{\min}(X)$ be **smallest p -local functorial retract** of $\Omega\Sigma X$ that contains X . Then there is a functorial decomposition $\Omega\Sigma X \simeq \Omega Q^{\max}(X) \times A^{\min}(X)$ such that

$$L_n(\bar{H}_*(X)) \subseteq H_*(\Omega Q^{\max}(X))$$

for n not-a-power of p . In other words, The primitives in $H_*(A^{\min}(X))$ only have tensor length of a power of p .

- For the exponent problem, it tells that only obstructions are given by iterated Whitehead products with length a power of p .