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CONFIGURATION SPACES AND THE EVALUATION MAPS

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This talk will consist of:

1. Configuration Spaces.

2. Evaluation Maps.

3. Applications to the Exponent Problem.

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1. Configuration Spaces

Let M be a manifold. The **ordered configuration space**

$$F(M,n)=\{(z_1,\ldots,z_n)\in M^n \mid z_i\neq z_j \text{ for } i\neq j\}.$$

The **unordered configuration space**

$$B(M,n)=F(M,n)/\Sigma_n.$$

Examples: $\pi_1(B(M,n))$ is the n-strand braid group over M. In particular, $B_n=\pi_1(B(\mathbb{R}^2,n))$ is the classical Artin braid group:

generators: $\sigma_1,\ldots,\sigma_{n-1}$;

relations: $\sigma_i\sigma_j=\sigma_j\sigma_i$ for $|i-j|\geq 2$, and $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$.

• Any links can be obtained from by closing up an Artin braid.

• The Lie algebra obtained from the descending central series of the Artin pure braid group $P_n=\pi_1(F(\mathbb{R}^2,n))$ is so-called Yang-Baxter Lie algebra.

• Vassiliev invariants can be obtained from the homology of the loop space of ordered configuration spaces by the works of Kohno and others.

• General higher homotopy groups of the sphere can be obtained by considering simplicial structure on braids by my joint works with Fred Cohen and Jon Berrick.

Let M_0 be a submanifold of M and let X be a pointed space. The **configuration space** $C(M, M_0; X)$ **with labels in** X is as follows:

$$C(M, M_0; X) = \coprod_{n=1}^{\infty} F(M, n) \times_{S_n} X^n / \approx$$

filtered by configuration length with

$$D_n(M, M_0; X) = C_n/C_{n-1} = F(M, n)/F(M|M_0, n) \wedge_{S_n} X^{(n)},$$

where $F(M|M_0, n)$ is the subspace of $F(M, n)$ consisting of all configurations with at least one coordinate in M_0, and $\approx$ is generated by

$$(z_1, \ldots, z_n; x_1, \ldots, x_n) \approx (z_1, \ldots, z_{n-1}; x_1, \ldots, x_{n-1})$$

if $z_n \in M_0$ or $x_n = *$.

Examples: $C(\mathbb{R};X)\simeq J(X)$ the James construction, the free monoid generated by the space X subject to the single relation $*=1$ with weak topology. If X is path-connected, then

$$C(\mathbb{R}^n;X)\simeq \Omega^n\Sigma^n X.$$

• In general, $C(M,M_0;X)$ is homotopy equivalent to the space of cross-sections of certain fibre bundle (with fibre $\Sigma^n X$) obtained from the tangent bundle of M if X or M/M_0 is path-connected.

• $C(M,S^0)=\coprod_{n=0}^{\infty} B(M,n)$, the disjoint union of unordered configuration spaces, which is a (homotopy) monoid if $M=N\times\mathbb{R}$.

2. Evaluation Maps

Recall that a map $f\colon \Sigma Y \to Z$ is called an *evaluation map* if its adjoint map $f'\colon Y \to \Omega Z$ is a (weak) homotopy equivalence. Let $I = [0,1]$. Consider the commutative diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{\ \sigma'\ } & C(M\times I, M_0; X) \\ \Big\downarrow q' & \text{pull-back} & \Big\downarrow q \simeq \Delta \\ \bigvee_{i=1}^{2} C(M_i, M_{i,0}; X) & \hookrightarrow & C(M\times I, M_0'; X), \end{array}$$

where q is a quasifibration,

(1). $M_0 = M \times ([0,1/5] \cup [4/5,1])$;

(2). $M_0' = M \times ([0,1/5] \cup [2/5,3/5] \cup [4/5,1])$;

(3). $M_1 = M \times [0,3/5]$;

(4). $M_{1,0} = M \times ([0,1/5] \cup [2/5,3/5])$;

(5). $M_2 = M \times [2/5,1]$;

(6). $M_{2,0} = M \times ([2/5,3/5] \cup [4/5,1])$;

There is an evaluation map

$$\sigma\colon \Sigma C(M\times I; X) \simeq A(X) \longrightarrow C(M\times I, M_0; X) \simeq C(M; \Sigma X)$$

with the following properties:

1) σ is functorial with respect to X.

2) σ preserves the configuration filtration and so it induces functorial maps

$$\bar{\sigma}_n\colon \Sigma D_n(M\times I;X)\longrightarrow D_n(M;\Sigma X).$$

3) Let

$$\bar{\psi}_n\colon D_n(M;\Sigma X)\longrightarrow \bigvee_{i=1}^{n-1} D_i(M;\Sigma X)\wedge D_{n-i}(M;\Sigma X)$$

be the induced map from the reduced diagonal $\bar{\Delta}_{C(M,\Sigma X)}$. Then the composite

$$\bar{\psi}_n\circ\bar{\sigma}_n\colon \Sigma D_n(M\times I;X)\longrightarrow \bigvee_{i=1}^{n-1} D_i(M;\Sigma X)\wedge D_{n-i}(M;\Sigma X)$$

is functorially null-homotopic.

Note: $A(X)$ is the union of two contractible spaces $C(M\times[0,3/5],M\times[0,1/5];X)$ and $C(M\times[2/5,1],M\times[4/5,1];X)$ with the intersection $C(M\times[2/5,3/5];X)$, which tells that

$$A(X)\simeq \Sigma C(M\times[2/5,3/5];X)\simeq \Sigma C(M\times I;X).$$

In particular, there is an evaluation map

$$\sigma\colon \Sigma C(\mathbb{R}^2;X)\simeq \Sigma C(I^2;X)\to C(\mathbb{R};\Sigma X)\simeq J(\Sigma X)$$

with the above three properties.

Theorem A (Desuspension Theorem). The evaluation maps $\bar{\sigma}_n\colon \Sigma D_n(M\times I;X)\to D_n(M;\Sigma X)$ is desuspensionable. A desuspension of $\bar{\sigma}_n$ can be given by the pinch map from $F(M\times I,n)$ to the quotient by the subspace

$$F((M\times I)|(M\times[0,\epsilon]),n)\cap F((M\times I)|(M\times[1-\epsilon,\epsilon]),n)$$

smashing with $X^{(n)}$ over S_n for small $\epsilon>0$.

• It seems that, by considering intersections of the subspaces $F((M\times I)|(M\times[t-\epsilon,t+\epsilon]),n)$, one may get a filtration on $F(M\times I,n)$ and $D_n(M\times I;X)$ built up by wedges of suspensions of $F(M,n)$ and $D_n(M;X)$, respectively.

3. Applications to the Exponent Problem

Barratt Conjecture: Let $f\colon \Sigma^2X\to Y$ such that $p^r[f]=0$ in the group $[\Sigma^2X,Y]$. Then

$$p^{r+1}\cdot \mathrm{Im}(f_*\colon \pi_*(\Sigma^2X)\to\pi_*(Y))=0.$$

For instance, the identity map of $\Sigma^2X\wedge P^n(p^r)$ satisfies the condition for $p^r\neq 2$, where $P^n(p^r)$ is the Moore space. (If $p^r=2$, then the identity map has order of 4 rather than 2.)

Cohen Groups: There is a progroup

$$\mathfrak{H} = \lim_{\overleftarrow{n}} \mathfrak{H}_n \longrightarrow \cdots \longrightarrow \mathfrak{H}_n \longrightarrow \mathfrak{H}_{n-1} \longrightarrow \cdots \longrightarrow \mathfrak{H}_1 = \mathbb{Z}$$

$\mathrm{Ker}(\mathfrak{H}_n \to \mathfrak{H}_{n-1}) = \mathrm{Lie}(n)$,

$\mathrm{Lie}(n) = \mathbb{Z}\{[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}, \ldots, x_{\sigma(n)}] \mid \sigma \in \Sigma_n\} \subseteq V_n^{\otimes n}$

and V_n is the free $\mathbb{Z}$-module with basis $\{x_1, \ldots, x_n\}$.

$\mathfrak{H}_n$ is the equalizer of the faces

$$d_i \colon K_n(x_1, \ldots, x_n) \to K_{n-1}(x_1, \ldots, x_{n-1})$$

for $1 \leq i \leq n$, where $d_i(x_i) = 1$, d_i maps other generators down in order, and K_n is the quotient of F_n subject to the relations

$$[[x_{i_1}, x_{i_2}], \ldots, x_{i_t}] = 1$$

if one of the generators occurs at least twice.

For any map $f \colon \Sigma^2 X \to Y$, there is a commutative diagram

$$\begin{array}{ccccc} \mathfrak{H} & \longrightarrow \cdots \longrightarrow & \mathfrak{H}_n & \longrightarrow & \cdots \\ \downarrow \theta_f & & \downarrow \theta_f & & \\ [J(\Sigma X), \Omega Y] & \longrightarrow \cdots \longrightarrow & [J_n(\Sigma X), \Omega Y] & \longrightarrow & \cdots \end{array}$$

• The image of θ_f is given by the composites

$$J(\Sigma X) \xrightarrow{\mathrm{Hopf}} J\left((\Sigma X)^{(n)}\right) \dashrightarrow J\left((\Sigma X)^{(n)}\right) \xrightarrow{\Omega W_n} J(\Sigma X) \xrightarrow{\Omega f} \Omega Y.$$

If $p^r[f]=0$, then the map $\theta_f\colon \mathfrak{H}_n\to[J_n(\Sigma X),\Omega Y]$ factors through the quotient $\mathfrak{H}_n^{\mathbb{Z}/p^r}$, where $\mathfrak{H}_n^{\mathbb{Z}/p^r}$ is the equalizer obtained from $K_n^{\mathbb{Z}/p^r}$ and $K_n^{\mathbb{Z}/p^r}$ is the quotient of K_n subject the second type relations:

$$x_i^{p^r}=1.$$

From

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \Sigma C_{n-1}(\mathbb{R}^2;X) & \longrightarrow & \Sigma C_n(\mathbb{R}^2;X) & \longrightarrow & \cdots & \longrightarrow & \Sigma C(\mathbb{R}^2;X) \\
& \downarrow \sigma_{n-1} & & \downarrow \sigma_n & & & & \downarrow \sigma \\
\cdots \longrightarrow & J_{n-1}(\Sigma X) & \longrightarrow & J_n(\Sigma X) & \longrightarrow & \cdots & \longrightarrow & J(\Sigma X),
\end{array}$$

there is a commutative diagram

$$\begin{array}{ccc}
\mathfrak{H}_n & \dashrightarrow & \mathfrak{R}_n \\
\downarrow \theta_f & & \downarrow \theta_f \\
[J_n(\Sigma X),\Omega Y] & \xrightarrow{\sigma_n^*} & [\Sigma C_n(\mathbb{R}^2;X),\Omega Y].
\end{array}$$

• Let $\mathfrak{R}$ be the quotient group of $\mathfrak{H}$ by adding so-called *shuffle relations*, which is essentially obtained from the reduced diagonal $J(\Sigma X)\to J(\Sigma X)\wedge J(\Sigma X)$. Then the composite $\mathfrak{H}\to[J(X),\Omega Y]\to[\Omega J(X),\Omega^2 Y]$ factors through $\mathfrak{R}$.

• $\mathfrak{R}$ is abelian. Moreover $\mathfrak{R}$ is a ring with multiplication induced from the composition of self maps of $\Omega\Sigma^2 X$. Thus the spaces $\Omega^2\Sigma^2 X$ can be regarded as a *module* over $\mathfrak{R}$ in the homotopy category.

• The group $\mathfrak{H}$ is isomorphic to the group of all functorial coalgebra maps of tensor algebras of projective modules over $\mathbb{Z}$. In general, for any commutative ring R, there is a progroup $\mathfrak{H}^R$ and its quotient $\mathfrak{R}^R$ which is pro-ring. When $R=\mathbb{Z}$, $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p^r$, the groups $\mathfrak{H}^R$ and $\mathfrak{R}^R$ admit geometric means as above.

• The representation theory of $\mathfrak{R}$ correspond to the functorial decompositions of loop suspensions and is related to the functorial version of the Poincaré-Birkhoff-Witt Theorem considered by Paul Selick and me.

• There is a morphism of rings

$$\theta\colon \mathfrak{R}^R \longrightarrow \prod_{n=1}^{\infty} \mathrm{Hom}_{R(S_n)}(\mathrm{Lie}^R(n), \mathrm{Lie}^R(n)).$$

If R is a field of characteristic 0, then θ is an isomorphism.

•**Theorem B.** Let $\mathfrak{J}^R_n$ denote the kernel of $\mathfrak{R}^R_n \to \mathfrak{R}^R_{n-1}$. Then there is an exact sequence

$$0 \longrightarrow \mathrm{Ext}_{R(S_n)}(\mathrm{Lie}^R(n)^*, \mathrm{Lie}^R(n)^*) \longrightarrow \mathfrak{J}^R_n$$

$$\xrightarrow{\theta} \mathrm{End}_{R(S_n)}(\mathrm{Lie}^R(n)) \longrightarrow H^0_{R(S_n)}(\mathrm{Lie}^R(n)^*; \mathrm{Lie}^R(n)^*) \longrightarrow 0$$

for each n.

For an $R(S_n)$-module M, write $M[-1]$ for the module M with signed S_n-action, that is, $M[-1] = M \otimes R[-1]$.

Theorem C. Let B_n act on $\mathrm{Lie}^R(n)$ via the canonical quotient $B_n \to S_n$. Then there is an isomorphism

$$\mathfrak{J}^R_n \cong H^{n-1}(B_n; \mathrm{Lie}^R(n)[-1]) \cong H^{n-1}(B_n; H_{n-1}(P_n; R))$$

where $P_n = \mathrm{Ker}(B_n \to S_n)$ is the Artin pure braid group with the canonical B_n-action on $H_*(P_n; R)$.

• This theorem gives some connections with the complexity of algorithms studied by Smale, C. De Concini, C. Procesi, M. Salvetti and other people.

★ The proof is given by handling the evaluation maps

$$\bar{\sigma}_n \colon \Sigma D_n(\mathbb{R}^2; X) \longrightarrow (\Sigma X)^{(n)}$$

and by looking at the equivariant pinch map

$$F(I^2,n) \longrightarrow F(I^2,n)/(F(I^2|I\times[0,\epsilon],n)\cap F(I^2|I\times[1-\epsilon,1],n)).$$

Example. Let $\mathrm{Lie}^{\max}(n)$ be the maximal $R(S_n)$-projective submodule of $\mathrm{Lie}(n)$, introduced by Paul Selick and me in the AMS Memoirs. Let $R=\mathbb{Z}_{(p)}$ or Z/p^r. According to Corollary 11.10 in that book, there is a decomposition

$$\mathrm{Lie}(p)=\mathrm{Lie}^{\max}(p)\oplus\mathrm{Lie}^{\min}(p)$$

with a short exact sequence of $R(S_p)$-modules

$$\mathrm{Lie}^{\min}(p)\hookrightarrow R^{\oplus p}\twoheadrightarrow R.$$

Thus in the stable homotopy category of $R(S_n)$-modules,

$$\mathrm{Lie}(p)\simeq\mathrm{Lie}^{\min}(p)\simeq\Sigma^{-1}R\Longrightarrow\mathrm{Lie}(p)^*\simeq\Sigma R$$

and so

$$\mathrm{Ext}_{R(S_p)}(\mathrm{Lie}^R(p)^*,\mathrm{Lie}^R(p)^*)\cong\mathrm{Ext}_{R(S_p)}(R,R)$$

the first cohomology of S_p, which is 0 if $R=\mathbb{Z}_{(p)}$ and $\mathbb{Z}/p$ if $R=\mathbb{Z}/p^r$ because $\mathrm{Syl}_p(S_p)=\mathbb{Z}/p$.