Connections between algebraic topology and the theory of braids

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Braids in Routine Life

Braids in Mathematics

Braids in Sciences

Our Mathematical Work
Braids in Routine Life

Braids in Routine Life

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Braids in Mathematics

Braids in Sciences

Our Mathematical Work

Braids in Routine Life
How to Make Braids
Find Mathematics: Basic Operation 1

\[
\begin{array}{cccccc}
1 & i-1 & i & i+1 & n \\
\sigma_i & & & & \\
\end{array}
\]
Find Mathematics: Basic Operation 2
Braid Groups

- Any $n$-strand braid is a product of $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and $\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}$.

- For instance, all 2-strand braids are given by the products of $\sigma_1, \sigma_1^{-1}$.

- For instance, all 3-strand braids are given by the products of $\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}$.

- All 4-strand braids are given by the products of $\sigma_1, \sigma_2, \sigma_3, \sigma_1^{-1}, \sigma_2^{-1}, \sigma_3^{-1}$.

- All 5-strand braids are given by the products of $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_1^{-1}, \sigma_2^{-1}, \sigma_3^{-1}, \sigma_4^{-1}$.
Example: What is the braid picture for the braid word \( \sigma_2\sigma_1^{-1}\sigma_3\sigma_2^2 \)?
Pure Braids $A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots \sigma_{i+1}\sigma_i \sigma_{i+1}^{-1}\cdots \sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$
Example: Discover formulas on braid words:

\[ \sigma_k A_{i,k+1} \sigma_k^{-1} = A_{i,k+1}^{-1} A_i A_{i,k+1}. \]
Brunnian Braids

A braid is called **Brunnian** if it becomes a trivial braid after removing **ANY** of its strands.

\[(\sigma_1 \sigma_2^{-1})^3\] is Brunnian
Discover $n$-strand Brunnian braids

Given any $(n-1)$-strand Brunnian braid $\beta$, $\sigma_{n-1}^{-2}\beta^{-1}\sigma_{n-1}^{2}\beta$ is an $n$-strand Brunnian braid.
Cabling a braid for having a braid with more strands
Cabling is an operation on braids by inserting parallel strands in a small neighbourhood of any strand of a braid.
Close-up braids to get links and knots

A link is the union of mutually disjoint simple closed (polygonal) curves in $\mathbb{R}^3$. A knot is a simple closed (polygonal) curve in $\mathbb{R}^3$. 
Braids and Polynomials

- Given an $n$-strand braid, by taking the intersection of the strands with the planes parallel to the $XY$-plane, one obtains a path
  \[(z_1(t), z_2(t), \ldots, z_n(t))\]
  with $z_i(t) \neq z_j(t)$ for each $0 \leq t \leq 1$ and any $i \neq j$, and so a polynomial
  \[f(x)(t) = (x - z_1(t))(x - z_2(t)) \cdots (x - z_n(t))\]
  with parameter $0 \leq t \leq 1$.

- From this, the braid group $B_n$ is the fundamental group of the space of all monic polynomials of degree $n$ with $n$ distinct roots.

- The braid groups are used to study complexity of approximating roots of polynomials.
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Scientific Meaning of Braids

A braid is a locus of points (or particles or robots or birds...) moving through time without colliding.

- air traffic control problem.
- satellites in sky.
- many robots working a place.
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A robot moving between $m$ obstacles gives an $m+1$ stranded braid

Robot Motion Planning
Applications of Braid Groups

Lectures at IMS conference June 2007

- **Braids and robotic** Robert Ghrist, UIUC, USA
- **Braids, twist, writhe, and solar activity** Mitch A Berger, UCL, UK
- **Coloring n-string braids and tangles and its application to molecular biology** Junalyn Navarra-Madsen, Texas WU, USA
- **Length-based cryptanalysis of the braid group and some applications** David Garber, Holon IT, Israel
Bio applications

Medical surgery planning
Molecule motions: ligand-docking
Importance of Configuration Pathways: protein-folding

insight into protein interactions & function may lead to better structure prediction algorithms

Misfolded proteins – e.g. BSE prion protein
One of Our Main Results

- The $n$-th homotopy group of the sphere is given by the quotient of the $(n + 1)$-strand Brunnian braid group over the sphere modulo the $(n + 1)$-strand Brunnian braid group over the disk for $n \geq 4$.

- The determination of the general homotopy groups is one of the central problems in the area of algebraic topology, and one of the most challenging problems in mathematics.

- This gives a deep connection between algebraic topology and the theory of braids.
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Topology is the philosophy of mathematics studying the global structure of mathematical objects.

End of Talk. Thank You!