Topology and Poincaré Conjecture

Jie Wu

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National University of Singapore

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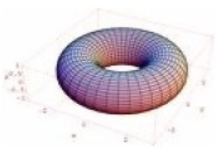
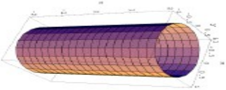
Surgery On Torus



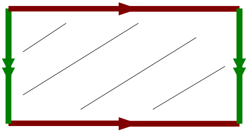
Cut it



Cut it once more

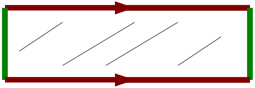
Correct Answer:

Torus

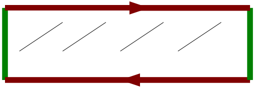


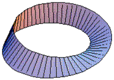
Discover New Surface

Cylinder



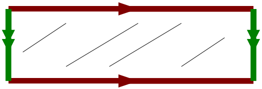
Mobius Band





Klein Bottle

Torus



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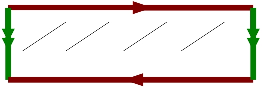

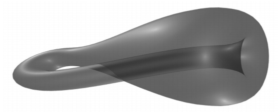

Topological Spaces

- Geometric objects like lines, planes, 3-dimensional spaces, spheres, torus, Mobiüs Band, Klein Bottle and etc are so-called **spaces**.
- Mathematical definition: A **topological space** means a set X with a family of subsets, so-called **open sets**, satisfying the property that the total set X and the empty set $\emptyset$ are open, the intersection of any two open sets is open, and an **arbitrary** union of open sets is open.
- You can learn **topological spaces** in the 4000 module, MA4266, Topology.
- Before taking this module, you can take Calculus modules (MA1104), Analysis modules (MA2108, MA3110), and Analysis on metric spaces (MA3209)
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Euclidean Spaces

- 0-dimensional Euclidean space $\mathbb{R}^0 = \{0\}$ is just a point.
- 1-dimensional Euclidean space $\mathbb{R}^1 = \mathbb{R}$ is a line.
- 2-dimensional Euclidean space $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ is a plane.
- $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is our 3-dimensional space.
- $\mathbb{R}^4 = \{(x, y, z, t) \mid x, y, z, t \in \mathbb{R}\}$ is our space-time, where t can be thought as time.
- In general, the n-dimensional Euclidean space $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$.
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Learning of Kids in School

- Mathematics: scoring x_1;
- Language: scoring x_2;
- Science: scoring x_3;
- Social Study: scoring x_4;
- Physical Education: scoring x_5;
- Music and Arts: scoring x_6;
- ....
- There are many variables: $x_1, x_2, x_3, x_4, x_5, x_6, \ldots$
- **My Conclusion:** Kids live in 6- or higher dimensional space. So they are very clever.

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n-dimensional Disks and Spheres

- The n-dimensional unit disk

$$D^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \leq 1\}$$

in $\mathbb{R}^n$.

- The n-dimensional unit sphere

$$S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sqrt{x_1^2 + x_2^2 + \cdots + x_{n+1}^2} = 1\}$$

consists of unit vectors in $\mathbb{R}^{n+1}$.

- $S^0 = \{-1, 1\}$, S^1 is the circle, S^2 is the standard sphere.
- **What does S^3 look like?**

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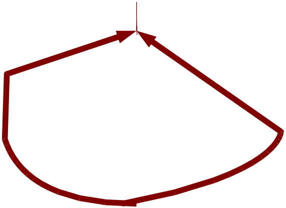
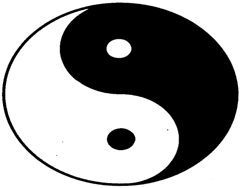
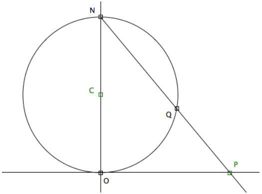
$S^1 = \mathbb{R} \cup \{\infty\}$

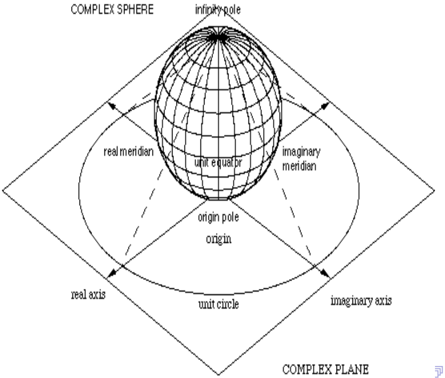
Future=Past





$S^2 = \mathbb{R}^2 \cup \{\infty\}$, $S^n = \mathbb{R}^n \cup \{\infty\}$





Manifolds

- An n-manifold is a space in which every point has a small open n-disk as its neighborhood.
- An n-manifold can be obtained by **gluing** n-disks together.
- The sphere S^2 is a 2-manifold (surface).
- The torus is a 2-manifold. The Klein Bottle is a 2-manifold.
- The Mobiüs Band is a 2-manifold **with boundary**.
- You can learn manifolds from the course MA5210, Differentiable Manifolds.

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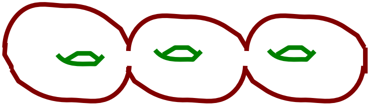
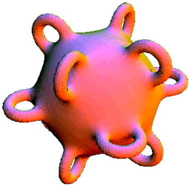
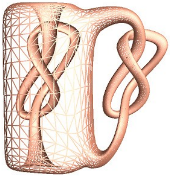
2-manifolds, Riemann Surfaces







Fundamental Group

- Given a space X, Fix point A. a **loop** means **continuous path** starting from A and ending with A.
- The **Fundamental Group**, denoted by $\pi_1(X)$, is the loops in X, where **two loops** are regarded as **the same**, called **homotopy**, if one can be **continuously deformed** into another.

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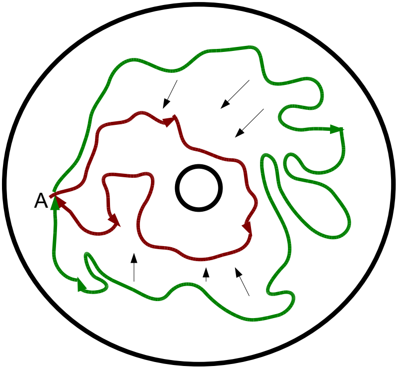
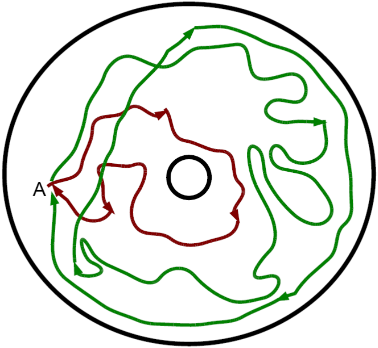
One Loop Can be deformed into Another

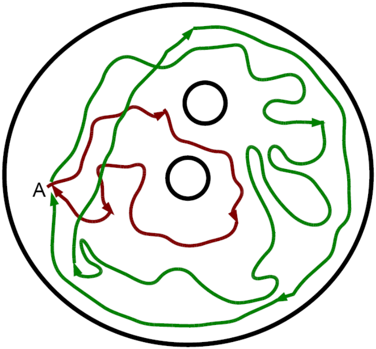


One Loop Can **NOT** be deformed into Another



GREEN goes TWICE around the hole
RED goes ONCE around the hole

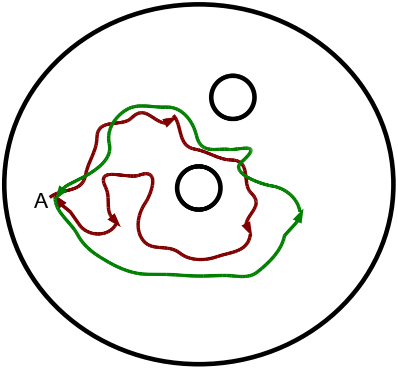
Product of loops: *ab* may not be the same as *ba*



RED*GREEN=Going along RED **first** and then GREEN.

GREEN*RED=Going along GREEN **first** and then RED.

Inverse of loop



Go Backwards.

Simply Connected Spaces

- A space X is **simply connected** if given any two points A and B there is a **continuous path** from A to B, and the **fundamental group** $\pi_1(X)$ is **trivial.**
- Namely **Any loop** can be continuously deformed into the **constant loop.**
- $S^2, S^3, S^4, \ldots,$ are simply connected.
- The circle S^1 is NOT simply connected.

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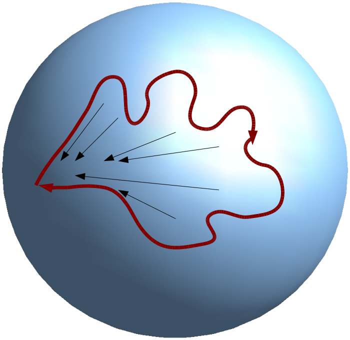
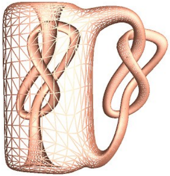
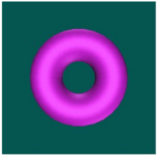
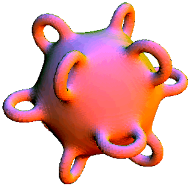
S^2 is simply connected

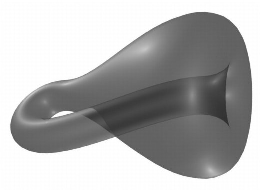


NOT simply connected spaces









Poincaré Conjecture

- ANY **simply-connected (closed)** 3-**manifold** must be a 3-sphere.
- In 1896, Poincaré was stating a **false theorem**: *Every compact polyhedral manifold with the homology of S^n is homeomorphic to S^n.* Then he found a beautiful counter example to his own claim. In 1904, he was asking whether every simply connected closed 3-manifold is homeomorphic to S^3. Since then, the above statement was known as Poincaré Conjecture .
- In 1934, as a part of a purported proof of PC, the famous topologist J. H. C. Whitehead gave a **false theorem**: *every open* 3*-manifold which is contractible (i.e. which can be continuously deformed to a point) is homeomorphic to* $\mathbb{R}^3$. Then he found a counterexample to his own claim.
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Poincaré Conjecture

- ANY **simply-connected (closed)** 3-**manifold** must be a 3-sphere.
- In 1896, Poincaré was stating a **false theorem**: *Every compact polyhedral manifold with the homology of S^n is homeomorphic to S^n.* Then he found a beautiful counter example to his own claim. In 1904, he was asking whether every simply connected closed 3-manifold is homeomorphic to S^3. Since then, the above statement was known as Poincaré Conjecture .
- In 1934, as a part of a purported proof of PC, the famous topologist J. H. C. Whitehead gave a **false theorem**: *every open* 3*-manifold which is contractible (i.e. which can be continuously deformed to a point) is homeomorphic to* $\mathbb{R}^3$*.* Then he found a counterexample to his own claim.
- The important counterexamples of Poincaré and Whitehead gave better understanding on Topology.

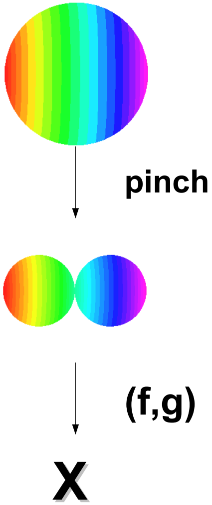
Progress on Poincaré Conjecture

- Generalized Poincaré Conjecture: A closed n-manifold of homotopy type of S^n is homeomorphic to S^n.
- Stephen Smale proved that this is true for $n \geq 5$ in 1960 for differentiable manifolds. Smale's proof was shorten by Andrew Wallace. Then John Stallings ($n \geq 7$) and Christopher Zeeman ($n = 5, 6$) proved for piecewise linear case.
- Micheal Freedman proved for $n = 4$.
- The actual Poincaré Conjecture ($n = 3$) was done recently: Grigori Perelman (sketched proof in 2003), Zhu Xiping and Cao Huaidong (complete proof in 2006).

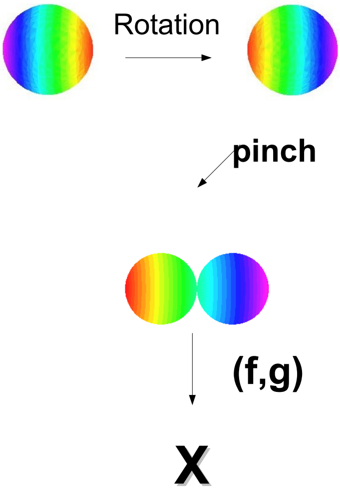
Homotopy Group

- Given a space X, Fix point A. Fix a point in S^n, say the north pole N in S^n.
- The *n*-**th homotopy group**, denoted by $\pi_n(X)$, is the continuous mapping f from S^n to X such that $f(N) = A$, where **two mappings** are regarded as **the same**, called **homotopy**, if one can be **continuously deformed** into another.
- When $n = 1$, this is the fundamental group.

Product $[f] + [g]$ in $\pi_n(X)$ for $n \geq 1$.



$[f] + [g] = [g] + [f]$ in $\pi_n(X)$ for $n \geq 2$.



Fundamental Problem in Algebraic Topology

- **Determine homotopy groups $\pi_n(X)$.**
- This problem is extremely open.
- Even if X is a sphere, $\pi_m(S^n)$ is **not yet well understood** for general m and $n \geq 2$, although many non-zero elements are known.
- There are a lot of mathematical problems related to homotopy groups.

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Why is it good to study Topology?

- By studying topology, you will have a **big room** for your future. If you have good results in topology, you may get highly recognized in the world. (A lot of people got Fields Medal due to their work on topology.) On the other hand, there are many accessible problems that you can just work on.
- There are a lot of applications of algebraic topology to computer science, physics and others. Some companies such as bank or computer companies actually like to hire Ph.D's in topology. Topology trains your **global views** on the world.

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Topology is the philosophy of mathematics studying the global structure of mathematical objects