

Homotopy Groups

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Topological Spaces

Homotopy

Classification Problem

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Topological Spaces and Continuous Maps

- From Calculus, a function $y = f(x)$ is continuous (in usual sense) **if and only if** the pre-image $f^{-1}(U)$ of any open set U is open.
- A **topological space** means a set X with a family of subsets, so-called **open sets**, satisfying the property that the total set X and the empty set \emptyset are open, the intersection of any two open sets is open, and an **arbitrary** union of open sets is open.
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- Namely f can be **continuously** deformed to be g .
- A space X is called **homotopy equivalent** to Y , denoted by $X \simeq Y$, if there exist (continuous) maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \simeq \text{identity}_X$ and $f \circ g \simeq \text{identity}_Y$.

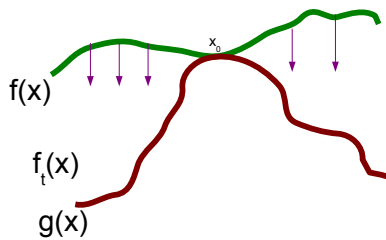
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Examples

- The identity map of the n -dimensional disk D^n is homotopic to the constant map.
- $D^n \simeq \text{point}$
- But the identity map of $D^n \setminus \{0\}$ is not homotopic to the constant map.
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Classification Problem in Homotopy Theory

- Given two (topological) spaces X and Y , how can we know whether $X \simeq Y$ or not?
- Algebraic topology studies the spaces using algebraic methods (groups, modules...), and can use these algebraic methods to **partially** answer this question.
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Definition of Homotopy Groups

- The n -dimensional unit sphere

$$S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sqrt{x_1^2 + x_2^2 + \dots + x_{n+1}^2} = 1\}$$

consists of unit vectors in \mathbb{R}^{n+1} with a choice of base-point $(-1, 0, \dots, 0)$.

- $\pi_n(X) := [S^n, X]$, the set of the (pointed) homotopy classes of (pointed) continuous maps from the n -sphere S^n to X .

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- $S^0 = \{-1, 1\}$.
- $f: S^0 = \{-1, 1\} \rightarrow X$ with $f(-1) = x_0 \iff f(1) = x \in X$ for any $x \in X$.
- $f \simeq g: S^0 \rightarrow X \iff$ there is a path from $f(1)$ to $g(1)$.
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Fundamental Group $\pi_1(X)$

- The loops λ from x_0 to x_0 with $\lambda \simeq \lambda'$ if λ can be deformed to λ' .
- $\pi_1(X)$ is a group with product given by path composition, which is not commutative in general.
- For any 2-manifolds (surfaces) X and X' , $X \simeq X' \Leftrightarrow \pi_1(X) \cong \pi_1(X')$
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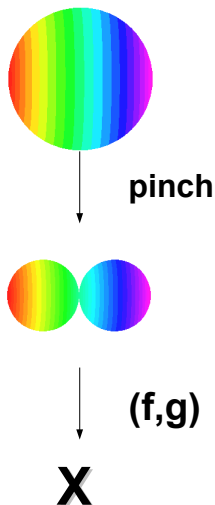
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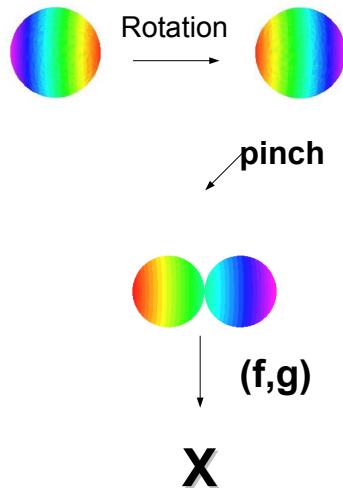
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Product $[f] + [g]$ in $\pi_n(X)$ for $n \geq 1$.



$[f] + [g] = [g] + [f]$ in $\pi_n(X)$ for $n \geq 2$.



Remarks

- Čech defined the higher homotopy groups, but abandoned them they are abelian. (1930s)
- It was originally conjectured that the homotopy groups of spheres are isomorphic to their homology groups. Then Heinz Hopf invented the Hopf map.
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Examples

- $\pi_n(S^1) = 0$ for $n \neq 1$ and $\pi_1(S^1) = \mathbb{Z}$.
- For $n > 0$, $\pi_m(S^n) = 0$ for $m < n$ and $\pi_n(S^n) = \mathbb{Z}$.
- Curtis proved that $\pi_i(S^5) \neq 0$ for all $i \geq 5$.
- $\pi_m(S^n)$ for $m > n$ is **not yet well understood** for general m and $n \geq 2$, although many non-zero elements are known.

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- Let ΩX be the space of **all continuous loops** from x_0 to x_0 .
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EHP sequences

- **EHP fibration:** $S^n \longrightarrow \Omega S^{n+1} \longrightarrow \Omega S^{2n+1}$ **localized at 2.**
- There is a long exact sequence (localized at 2)

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- Toda did computations on the homotopy groups of the spheres using EHP sequences in 1960s. Toda's table is available in my web-site.
- Later people have been to use Adams spectral sequences for computing the homotopy groups (of the spheres).
- People also have been to study a family of elements in the homotopy groups.
- People have been to study **properties** of homotopy groups. Cohen-Moore-Neisendorfer Theorem: for $p > 2$

$$p^n \cdot (p - \text{torsion component of } \pi_*(S^{2n+1})) = 0.$$

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Combinatorial Descriptions of Homotopy Groups

- Let $G(n)$ be the group generated by x_1, x_2, \dots, x_n with the relations:
 - (1) the product $x_1 x_2 \cdots x_n = 1$ and
 - (2) all iterated commutators

$$[[x_{i_1}^{\epsilon_1}, x_{i_2}^{\epsilon_2}], \dots, x_{i_t}^{\epsilon_t}] = 1,$$

$\epsilon_j = \pm 1$, if **each** x_j occurs **at least once** in the bracket.

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- Recently we are able to obtain the EHP sequences for $(p - 1)$ -cell complexes. (Generalize James' EHP sequences for spheres given 50 years ago.)
- There are a lot of problems related to homotopy groups. Some problems are extremely difficult, while others are pretty accessible for Ph.D. thesis.

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