Homotopy Groups

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Topological Spaces and Continuous Maps

- From Calculus, a function $y = f(x)$ is continuous (in usual sense) **if and only if** the pre-image $f^{-1}(U)$ of any open set U is open.
- A **topological space** means a set X with a family of subsets, so-called **open sets**, satisfying the property that the total set X and the empty set $\emptyset$ are open, the intersection of any two open sets is open, and an **arbitrary** union of open sets is open.
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- Let $f, g\colon X \to Y$ are two (continuous) maps with $f(x_0) = g(x_0)$. Call f is homotopic to g (relative to x_0), denoted by $f \simeq g$, if there exists $f_t(x)\colon X \to Y$ with parameter $0 \leq t \leq 1$ such that $f_t(x)$ is **continuous** on x and t, $f_0 = f$, $f_1 = g$, and $f_t(x_0) = f(x_0) = g(x_0)$.
- Namely f can be **continuously** deformed to be g.
- A space X is called **homotopy equivalent** to Y, denoted by $X \simeq Y$, if there exist (continuous) maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{identity}_X$ and $f \circ g \simeq \mathrm{identity}_Y$.

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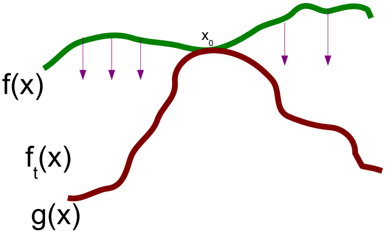

Examples

- The identity map of the n-dimensional disk D^n is homotopic to the constant map.
- $D^n \simeq \text{point}$
- But the identity map of $D^n \setminus \{0\}$ is not homotopic to the constant map.
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Classification Problem in Homotopy Theory

- Given two (topological) spaces X and Y, how can we know whether $X \simeq Y$ or not?
- Algebraic topology studies the spaces using algebraic methods (groups, modules...), and can use these algebraic methods to **partially** answer this question.
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Definition of Homotopy Groups

- The n-dimensional unit sphere

  $$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sqrt{x_1^2 + x_2^2 + \cdots + x_{n+1}^2} = 1\}$$

  consists of unit vectors in $\mathbb{R}^{n+1}$ with a choice of base-point $(-1, 0, \ldots, 0)$.
- $\pi_n(X) := [S^n, X]$, the set of the (pointed) homotopy classes of (pointed) continuous maps from the n-sphere S^n to X.

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$\pi_0(X)$

- $S^0 = \{-1, 1\}$.
- $f \colon S^0 = \{-1, 1\} \to X$ with $f(-1) = x_0 \iff f(1) = x \in X$ for any $x \in X$.
- $f \simeq g \colon S^0 \to X \Leftrightarrow$ there is a path from $f(1)$ to $g(1)$.
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Fundamental Group $\pi_1(X)$

- The loops λ from x_0 to x_0 with $\lambda \simeq \lambda'$ if λ can be deformed to λ' .
- $\pi_1(X)$ is a group with product given by path composition, which is not commutative in general.
- For any 2-manifolds (surfaces) X and X', $X \simeq X' \Leftrightarrow \pi_1(X) \cong \pi_1(X')$
- For any closed 2-manifolds (surfaces) X and X', $X \cong X' \Leftrightarrow \pi_1(X) \cong \pi_1(X')$

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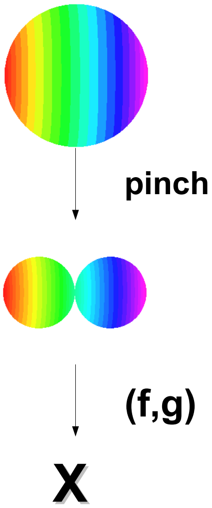
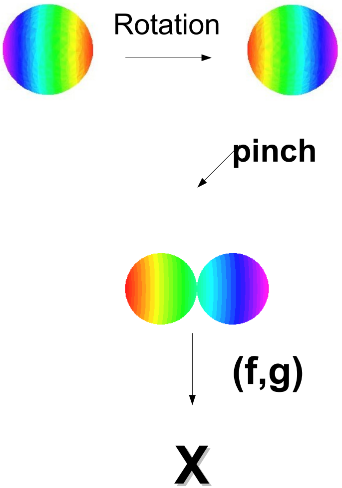
Product $[f] + [g]$ in $\pi_n(X)$ for $n \geq 1$.



$[f] + [g] = [g] + [f]$ in $\pi_n(X)$ for $n \geq 2$.



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- Čech defined the higher homotopy groups, but abandoned them they are abelian. (1930s)
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Examples

- $\pi_n(S^1) = 0$ for $n \neq 1$ and $\pi_1(S^1) = \mathbb{Z}$.
- For $n > 0$, $\pi_m(S^n) = 0$ for $m < n$ and $\pi_n(S^n) = \mathbb{Z}$.
- Curtis proved that $\pi_i(S^5) \neq 0$ for all $i \geq 5$.
- $\pi_m(S^n)$ for $m > n$ is **not yet well understood** for general m and $n \geq 2$, although many non-zero elements are known.

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Why are mapping spaces useful?

- Let ΩX be the space of **all continuous loops** from x_0 to x_0.
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EHP sequences

- EHP **fibration**: $S^n \longrightarrow \Omega S^{n+1} \longrightarrow \Omega S^{2n+1}$ **localized** at 2.
- There is a long exact sequence (localized at 2)

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- Later people have been to use Adams spectral sequences for computing the homotopy groups (of the spheres).
- People also have been to study a family of elements in the homotopy groups.
- People have been to study **properties** of homotopy groups. Cohen-Moore-Neisendorfer Theorem: for $p > 2$

$$p^n \cdot (p - \text{torsion component of} \pi_*(S^{2n+1})) = 0.$$

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Combinatorial Descriptions of Homotopy Groups

- Let $G(n)$ be the group generated by $x_1, x_2, \ldots, x_n$ with the relations:
  - (1) the product $x_1 x_2 \cdots x_n = 1$ and
  - (2) all iterated commutators

$$[[x_{i_1}^{\epsilon_1}, x_{i_2}^{\epsilon_2}], \ldots, x_{i_t}^{\epsilon_t}] = 1,$$

  $\epsilon_i = \pm 1$, if **each** x_i occurs **at least once** in the bracket.
- $\pi_n(S^2) = Z(G(n))$, the center of $G(n)$.
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- It is **NOT** straightforward to know the homotopy groups of finite cell complexes by assuming the homotopy groups of the spheres.
- Recently we are able to obtain the EHP sequences for $(p-1)$-cell complexes. (Generalize James' EHP sequences for spheres given 50 years ago.)
- There are a lot of problems related to homotopy groups. Some problems are extremely difficult, while others are pretty accessible for Ph.D. thesis.

Advantage for studying Topology

- By studying algebraic topology or homotopy theory, you will have a **big room** for your future. If you have good results in homotopy theory, you may get highly recognized in the world. On the other hand, there are many accessible problems that you can just work on.
- There are a lot of applications of algebraic topology to computer science, physics and others. Some companies such as bank or computer companies actually like to hire Ph.D's in topology. Topology trains your **global views** on the world.

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