Adams Spectral Sequences and the Yang-Baxter Lie algebra

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Spectral Sequences for homology Theory

- Any cofibre sequence $A \to X \to Z$, that is, $Z$ is homotopy equivalent to the mapping cone of $A \to X$, induces a long exact sequence for any homology theory.
- Given a space $X$, a sequence of cofibrations
  \[ X_1 \to X_2 \to X_3 \to \cdots \to X_n \to \cdots \to X \]
  induces a spectral sequence on homology. Under certain conditions, such as the homology of $X$ is the (direct) limit of the homology of $X_n$ and other conditions, the spectral sequence may converge to the homology of $X$.
- Filtration on the total space of a fibration $\implies$ Leray-Serre Spectral Sequences.
- Skeleton filtration on $CW$-complexes $\implies$ Atiyah-Hirzebrauch-Whitehead Spectral Sequences.
Spectral Sequences for homotopy groups

- Key point: a **fibration** induces a long exact sequence of homotopy groups.

- Given a space $X$, sequence of fibrations

  $$\cdots \to X_n \to \cdots \to X_2 \to X_1 \to X$$

  $$X \to \cdots \to X_n \to \cdots \to X_2 \to X_1$$

  induce a spectral sequence on homotopy groups. Under certain conditions, the spectral sequence may **converge** to the homotopy groups of $X$.

- Adams Spectral Sequences: **Resolutions** with respect to homology theory. **Philosophically** it is similar to projective resolutions of modules.
Definition

- A simplicial set means a $\Delta$-set $X$ together with a collection of degeneracies $s_i : X_n \to X_{n+1}, 0 \leq i \leq n$, such that
  \[ d_j d_i = d_{i-1} d_j \]  
  for $j < i$,  
  \[ s_j s_i = s_{i+1} s_j \]  
  for $j \leq i$ and  
  \[ d_j s_i = \begin{cases} s_{i-1} d_j & j < i \\ \text{id} & j = i, i+1 \\ s_i d_{j-1} & j > i + 1. \end{cases} \]  
  The three identities for $d_i d_j$, $s_j s_i$ and $d_i s_j$ are called the simplicial identities.

- A simplicial map $f$ means a sequence of functions $f : X_n \to Y_n$ such that $d_i f = f d_i$ and $s_i f = f s_i$.

- One can use deleting-doubling coordinates for catching simplicial identities.
Definition and Remarks

• Let $\mathcal{C}$ be a category. A *simplicial object over* $\mathcal{C}$ means a sequence of objects $X = \{X_n\}_{n \geq 0}$ with face morphisms $d_i : X_n \to X_{n-1}$ and degeneracy morphisms $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$, such that the three simplicial identities hold.

• In particular, one has the notion of simplicial groups.

• The homotopy category of $CW$-complexes is equivalent to the homotopy category of simplicial sets.

• Any loop space is (weakly) homotopy equivalent to the geometric realization of a simplicial group.
• Let $H$ be a simplicial subgroup of a simplicial group $G$. Then

$$H \hookrightarrow G \rightarrow G/H$$

is a (simplicial) fibration.

• Given a simplicial group $G$, a filtration of simplicial subgroups

$$\cdots \subseteq G^n \subseteq \cdots \subseteq G^2 \subseteq G^1 \subseteq G$$

induces a spectral sequence (converging to $\pi_\ast(G)$ under some assumptions).

• Different simplicial groups may have the same homotopy type. So they may induce different spectral sequences.

• Fix a simplicial group. Different filtrations may induce different spectral sequences.
Milnor’s Construction

- A pointed set $S$ means a set $S$ with a basepoint $\ast$. Denote by $F[S]$ the free group generated by $S$ subject to the relation $\ast = 1$.
- Let $X$ be a pointed simplicial set. Milnor’s construction is the simplicial group $F[X]$, where $F[X]_n = F[X_n]$ with face and degeneracy homomorphisms induced by the faces and degeneracies of $X$.
- **Universal Property:** For any simplicial group $G$ and any pointed simplicial map $f : X \rightarrow G$, there is a unique simplicial homomorphism $\tilde{f} : F[X] \rightarrow G$ such that $\tilde{f}|_X = f$.
- **Theorem.** $|F[X]| \simeq \Omega \Sigma |X|$. The loop space of the suspension of $|X|$.
- **James’ Construction.** $J(X)$ is the free monoid generated by $X$ subject to the relation $\ast = 1$.
- **Theorem.** If $|X|$ is path-connected, then the inclusion $|J(X)| \rightarrow |F[X]|$ is a homotopy equivalence.
Some Canonical Filtrations on (Simplicial) Groups

- Descending Central Series of a (simplicial) group
  \[
  \cdots \subseteq \Gamma^n G \subseteq \cdots \subseteq \Gamma^3 G \subseteq \Gamma^2 G \subseteq \Gamma^1 G = G,
  \]
  where \( \Gamma^2 G = [G, G] \) the commutator subgroup, 
  \( \Gamma^{n+1} G = [\Gamma^n G, G] \).

- If \( G \) is a free group, then
  \[
  \text{Gr.}(G) = \bigoplus_{n=1}^{\infty} \Gamma^n G/\Gamma^{n+1} G
  \]
  is the free Lie algebra generated by \( G/[G, G] \).

- Mod \( p \) descending central series of Milnor’s construction \( F[X] \) induces the Adams-Bousfield-Kan spectral sequence converging to \( \pi_*(F[X]) = \pi_*(\Omega \Sigma |X|) = \pi_{*+1}(\Sigma |X|) \).

- There are other potential spectral sequences such as Burnside subgroups filtration of a simplicial group.
Simplicial Structure on Braids

- The sequence of Artin braid groups \( \mathcal{B} = \{B_{n+1}\}_{n \geq 0} \) is a crossed simplicial group with the faces and degeneracies described as follows:
  
  Given an \((n+1)\)-strand braid \( \beta \in B_{n+1} \), \( d_i \beta \) is obtained by removing \((i+1)\)st strand braid and \( s_i \beta \) is obtained by doubling \((i+1)\)-strand for \( 0 \leq i \leq n \), where the strands are counted from initial points.

- Since the restriction of \( d_i \) and \( s_i \) to the pure braid groups are group homomorphisms, the sequence of groups \( \text{AP}_* = \{P_{n+1}\} \) is a simplicial group. Note that \( \text{AP}_0 = P_1 \) is the trivial group.

- There is a simplicial map \( f : S^1 \to \text{AP}_* \) which sends the non-degenerate 1-simplex of \( S^1 \) to the generator \( A_{1,2} \) of \( \text{AP}_1 = P_2 \).

- The simplicial map \( f : S^1 \to \text{AP}_* \) extends uniquely to a simplicial homomorphism \( \Theta : F[S^1] \to \text{AP}_* \). 
Questions on $\Theta: F[S^1] \rightarrow \mathbb{A}P_*$

- $|F[S^1]| \simeq \Omega S^2$.
- $F[S^1]_n$ is a free group of rank $n$.
- Is $\Theta: F[S^1] \rightarrow \mathbb{A}P_*$ a monomorphism? The answer is Yes. (Joint with Fred Cohen.)
- How does it related to the Adams spectral sequence?
- We (with F. Cohen) considered the descending central series for answering the first questions.
- Further connections with the Adams spectral sequences leave you to explore.
Yang-Baxter Lie algebras

- $G = P_k(\mathbb{R}^2) = P_k$.
- Recall Artin’s generators $A_{i,j}$ for $P_k$, $1 \leq i < j \leq k$.
- $A_{i,j}$ projects to the elements $B_{i,j}$ in $P_k/[P_k, P_k]$.
- $\text{Gr.}(P_k) = \bigoplus_{n=1}^{\infty} \Gamma^n P_k / \Gamma^{n+1} P_k$ is the Lie algebra generated by $B_{i,j}$ subject to the following “horizontal 4T relations” or “infinitesimal braid relations”:
  1. $[B_{i,j}, B_{s,t}] = 0$ if $\{i, j\} \cap \{s, t\} = \emptyset$,
  2. $[B_{i,j}, B_{i,t} + B_{t,j}] = 0$ for $1 \leq j < t < i \leq k$, and
  3. $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ for $1 \leq j < t < i \leq k$.

- The above theorem was proven by T. Kohno and subsequently by M. Falk and R. Randell (in a context which also applies to certain other choices of $\text{Gr.}(G)$ for a large family of groups $G$).
Theorem

• **Theorem (Cohen-Wu):** The induced map of Lie algebras

\[
\text{Gr.}(\Theta_n) : \text{Gr.}(\mathcal{F}_n) \to \text{Gr.}(\mathcal{P}_{n+1})
\]

is a monomorphism.
• The proof follows from a horrible computation.
• It follows that the map

\[
\Theta_n : \mathcal{F}_n \to \mathcal{P}_{n+1}
\]

is a monomorphism, and so \(\mathcal{F}[S^1]\) embeds into \(\text{AP}_*\).
• \(E^0\)-terms of the Adams spectral sequence for \(\mathcal{F}[S^1]\) is a simplicial free (restricted) Lie algebra.
• \(E^1\)-terms are given by the \(\Lambda\)-algebra.
• \(E^2\)-terms are given by the extension group over the Steenrod algebra.