FUNCTIONAL HOMOTOPY DECOMPOSITIONS OF LOOPED CO-H SPACES

PAUL SELICK, STEPHEN THERIAULT AND JIE WU†

Abstract. In recent work of the first and third authors, functorial coalgebra decompositions of tensor algebras were geometrically realized to give functorial homotopy decompositions of loop suspensions. Later work by all three authors generalized this to functorial decompositions of looped coassociative co-H spaces. In this paper we use different methods which allow for the coassociative hypothesis to be removed.

1. Introduction

In studying any mathematical object it is useful to decompose it into irreducible components, analyze the components, and assess how they assemble to give the original object. A natural collection of topological spaces to study in this manner are H-spaces, as the multiplication provides an operation which allows for decompositions (up to homotopy). In practise, this requires a delicate interplay between algebra and topology. If X is an H-space, a decomposition $X \simeq A \times B$ will usually be an equivalence of spaces rather than H-spaces. This is reflected in a corresponding coalgebra decomposition $H_*(X) \cong H_*(A) \otimes H_*(B)$. However, a coalgebra decomposition $H_*(X)$ need not correspond to a decomposition of X on the level of spaces. Nevertheless, in attempting to decompose X, one first tries to find a coalgebra decomposition of $H_*(X)$ and then geometrically realize it. To decompose in homology it is advantageous to have a Kunneth isomorphism, so we take homology with field coefficients. On the level of spaces, this corresponds to looking for localized homotopy decompositions. So from now on we assume that all spaces and maps have been localized at a prime p and take homology with mod-p coefficients.

Using homology decompositions to predict homotopy decompositions has been a fruitful approach. The great success story was Cohen, Moore, and Neisendorfer's [5, 6] determination of the odd primary homotopy exponents of spheres by decomposing loop spaces related to the loops on mod-p^r Moore spaces. It has also proved useful in examining many other loop spaces [1, 4, 7, 9, 10, 12, 13, 14, 15, 19, 20, 21]. In all these cases the object was to decompose specific examples of loop spaces, often into indecomposable summands. The question arose whether, instead of working on a case-by-case basis, one could produce decompositions in a more general manner. This was achieved by the first and third authors [17, 18], who used connections between topology and the modular representation of the symmetric group to obtain *functorial* decompositions of loop suspensions. This approach has the advantage of producing decompositions for all loop suspensions simultaneously, although

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for a given loop suspension it may be possible to refine the functorial decomposition by using non-functorial information specific to the space. One such decomposition obtains the minimal functorial homotopy retract of a loop suspension. All such decompositions are the geometric realizations of functorial coalgebra decompositions of tensor algebras. The connection between tensor algebras and loop suspensions is given by the Bott-Samelson theorem, which states that $H_*(\Omega\Sigma X) \cong T(\bar{H}_*(X))$.

More generally, a simply-connected co-H space Y also satisfies $H_*(\Omega Y) \cong T(\Sigma^{-1}\bar{H}_*(Y))$, so one would guess that the functorial coalgebra decompositions of tensor algebras ought to translate into functorial decompositions of looped co-H spaces. In [16] this was shown to be true for coassociative co-H spaces based on the fact that the co-H structure map $Y \longrightarrow \Sigma\Omega Y$ is a co-H map, which allowed one to geometrically construct certain maps of spaces that were idempotents. In this paper we show that the coassociativity hypothesis can be removed, essentially by showing that it is enough to have maps which induce idempotents in homology. The removal of the coassociative hypothesis is significant. It is easy to produce examples of co-H spaces, as any retract of a suspension is a co-H space. On the other hand, it is usually difficult to determine when a co-H space is coassociative.

Our first result states that any functorial coalgebra decomposition of a tensor algebra can be geometrically realized as a functorial homotopy decomposition of looped co-H spaces. It generalizes [17, 18] which geometrically realized functorial homotopy decompositions of loop suspensions. Let V be a vector space over a field $\mathbf{k}$ and let $T(V)$ be the tensor algebra generated by V. Then $T(V)$ becomes a Hopf algebra by letting the elements of V be primitive and extending the coalgebra structure multiplicatively. Also, $T(V)$ has a grading by tensor length, and becomes bigraded in the case where V is graded. Let **CoH** be the category of co-H spaces and co-H maps.

Theorem 1.1 (Geometric Realization Theorem). *Let Y be any simply connected co-H-space of finite type and let*

$$T(V) \cong A(V) \otimes B(V)$$

be any natural coalgebra decomposition for ungraded modules over $\mathbb{Z}/p$. Then there exist homotopy functors $\bar{A}$ and $\bar{B}$ from **CoH** *to spaces such that:*

(1) *there is a functorial decomposition*
$$\Omega Y \simeq \bar{A}(Y) \times \bar{B}(Y);$$
(2) *in mod-p homology the decomposition*
$$H_*(\Omega Y) \cong H_*(\bar{A}(Y)) \otimes H_*(\bar{B}(Y))$$
is with respect to the augmentation ideal filtration;
(3) *in mod-p homology*
$$E^0 H_*(\bar{A}(Y)) = A(\Sigma^{-1}\bar{H}_*(Y)) \text{ and } E^0_*(\bar{B}(Y)) = B(\Sigma^{-1}\bar{H}_*(Y)),$$
where A and B are the canonical extensions of the functors A and B for graded modules.

One application of Theorem 1.1 is to obtain the minimal functorial homotopy retract of a looped co-H space, generalizing the result for loop suspensions in [18]. To state this precisely we require some notation. The algebraic functor $A^{\min}$ takes $\mathbf{k}$-modules to $\mathbf{k}$-coalgebras such that $A^{\min}(V)$ is the smallest natural coalgebra retract of $T(V)$. It inherits a (bi)grading from $T(V)$. Let $L_n(V)$ be the set of homogeneous Lie elements of tensor length n in the tensor algebra $T(V)$. Let $\bar{V}$ be the n-dimensional vector space with basis $\{x_1, x_2, \cdots, x_n\}$. Let $\mathrm{Lie}(n)$ be the submodule of $\bar{V}^{\otimes n}$ generated by the n-fold commutators from left to right

$$[[x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(n)}]$$

for σ in the symmetric group S_n. Let S_n act on Lie(n) by permuting the variables. Let $\mathrm{Lie}^{\max}(n)$ be the maximal projective $\mathbf{k}(S_n)$-submodule of Lie(n). For any $\mathbf{k}$-module V, let

$$L_n^{\max}(V) = V^{\otimes n} \otimes_{\mathbf{k}(S_n)} \mathrm{Lie}^{\max}(n).$$

Then $L_n^{\max}(V)$ is a functorial submodule of $L_n(V)$. Let $B^{\max}(V)$ be the sub Hopf algebra of $T(V)$ generated by $\{L_n^{\max}(V)\}_{n=2}^{\infty}$. By [17], there is a functorial coalgebra decomposition $T(V) \cong A^{\min}(V) \otimes B^{\max}(V)$. Theorem 1.1 therefore immediately gives the following.

Theorem 1.2. *There exist homotopy functors $\bar{A}^{\min}$ and $\bar{B}^{\max}$ from* **CoH** *to spaces such that for any p-local simply connected co-H space Y of finite type the following hold:*

(1) *$\bar{A}^{\min}(Y)$ is a functorial retract of ΩY and so there is a functorial decomposition*

$$\Omega Y \simeq \bar{A}^{\min}(Y) \times \bar{B}^{\max}(Y);$$

(2) *in mod-p homology the decomposition*

$$H_*(\Omega Y) \cong H_*(\bar{A}^{\min}(Y)) \otimes H_*(\bar{B}^{\max}(Y))$$

is with respect to the augmentation ideal filtration;

(3) *in mod-p homology*

$$E^0 H_*(\bar{A}^{\min}(Y)) = A^{\min}(\Sigma^{-1}\bar{H}_*(Y))$$
$$E^0 H_*(\bar{B}^{\max}(Y)) = B^{\max}(\Sigma^{-1}\bar{H}_*(Y)).$$

□

We give two examples to illustrate how Theorems 1.1 and 1.2 give new results. First, let $S^{2p} \xrightarrow{\alpha_1} S^3$ represent the generator of $\pi_{2p}(S^3) = \mathbb{Z}/p\mathbb{Z}$, which is the least nonvanishing p-torsion homotopy group of S^3. Let Y be its homotopy cofiber. By [2], Y is a p-local co-H space which is not homotopy coassociative. Therefore the results of [16] do not apply and so the decompositions of ΩY in Theorems 1.1 and 1.2 are new. Second, as mentioned, any retract Y of a suspension ΣX is a co-H space, so Theorems 1.1 and 1.2 can be applied to Y without any need to check for coassociativity as in [16]. An example of wide interest is the p-local decomposition $\Sigma\mathbb{C}P^n \simeq \bigvee_{i=1}^{p-1} Y_i$, where $H_*(Y_i)$ consists of those elements in $H_*(\Sigma\mathbb{C}P^n)$ in degrees of the form $2i+1+2j(p-1)$ for some $j \geq 0$. Each space Y_i is a co-H space, but it is unclear whether it is homotopy coassociative, and it would unpractical to check if this property holds using known methods.

The method for proving Theorem 1.1 is described briefly as follows. Let Y be any simply connected co-H-space. Let $\sigma\colon \Sigma\Omega Y \longrightarrow Y$ be the evaluation map. The comultiplication $Y \longrightarrow Y \vee Y$ induces a (unique up to homotopy) cross-section $s\colon Y \longrightarrow \Sigma\Omega Y$. To obtain functorial decompositions of ΩY, take homotopy colimits of composites of the form

$$\Omega Y \xrightarrow{\Omega s} \Omega\Sigma\Omega Y \xrightarrow{f_{\Omega Y}} \Omega\Sigma\Omega Y \xrightarrow{\Omega\sigma} \Omega Y$$

where the maps $f_{\Omega Y}$ induce idempotents in homology that are known to exist from the functorial the decompositions of loop suspensions in [18].

There are homological issues that need to be dealt with in doing all this. These stem from the fact that the isomorphism $H_*(\Omega Y) \cong T(\Sigma^{-1}\bar{H}_*(Y))$ is an isomorphism of algebras in general, not Hopf algebras. More precisely, the coalgebra structure on $T(\Sigma^{-1}\bar{H}_*(Y))$ is defined by letting the generators in $\Sigma^{-1}H_*(Y)$ be primitive. This is guaranteed to match the coalgebra structure on $H_*(\Omega Y)$ only when Y is a double suspension. When Y is a single suspension, or more generally a coassociative co-H space, then [3] is able to give formulas that measure the deviation from

$H_*(\Omega Y)\cong T(\Sigma^{-1}\bar{H}_*(Y))$ being a Hopf algebra isomorphism. If Y fails to be coassociative, then the coalgebra structure on $H_*(\Omega Y)$ may be very complex. To get around this, we filter $H_*(\Omega Y)$ by the augmentation ideal filtration. The corresponding associated graded module $E^0(H_*(\Omega Y))$ is isomorphic as a Hopf algebra to $T(\Sigma^{-1}\bar{H}_*(Y))$. In particular, on the level of associated graded modules, one has all the algebraic decompositions arising from the Poincaré-Birkhoff-Witt Theorem and other associated natural representations.

A filtered approach was also taken in [18] in the case of loop suspensions. However, the arguments there were directed solely towards obtaining the minimal functorial homotopy retract of a loop suspension rather than obtaining arbitrary functorial retracts of loop suspensions as in Theorem 1.1. So in this paper we begin in Section 2 with a recapitulation of the filtration ideas in [18], phrased in general terms which will allow us to obtain functorial homotopy retracts of looped co-H spaces. In Section 3 these will be used to prove Theorem 1.1.

2. Filtered Coalgebras and the Algebraic James Construction

2.1. **Filtered Coalgebras.** In this section, the ground ring is a field $\mathbf{k}$. A *coalgebra* shall refer to a pointed cocommutative graded coalgebra, and a *filtration* of a module M shall mean a decreasing filtration

$$\cdots\subseteq I^nM\subseteq I^{n-1}M\subseteq\cdots\subseteq I^0M=M.$$

For filtered modules M and N the product filtration on $M\otimes N$ is defined by

$$I^t(M\otimes N)=\sum_{a+b\geq t} I^aM\otimes I^bN.$$

Given a filtered module M, the tensor product $M^{\otimes n}$ becomes a filtered module by means of the product filtration. The ground ring $\mathbf{k}$ is filtered by $I^0\mathbf{k}=\mathbf{k}$ and $I^1\mathbf{k}=0$. A pointed *filtered coalgebra* C means a filtered module C with a filtered comultiplication $\psi\colon C\longrightarrow C\otimes C$ turning C into an augmented coalgebra with a filtered unit $\eta\colon\mathbf{k}\longrightarrow C$ and filtered augmentation $\epsilon\colon C\longrightarrow\mathbf{k}$. Let C and D be pointed filtered coalgebras. The wedge $C\vee D$ is defined by the pushout $C\longleftarrow\mathbf{k}\longrightarrow D$ in the category of cocommutative pointed coalgebras. The smash product $C\wedge D$ is defined to be the cokernel of $C\vee D\longrightarrow C\otimes D$. Let C be a filtered coalgebra. Define

$$E^0C=\bigoplus_{s=0}^{\infty} I^sC/I^{s+1}(C).$$

Let C be a pointed filtered coalgebra and let A be a filtered Hopf algebra. We write $[C,A]$ for the set of pointed filtered coalgebra morphisms from C to A. Let $f,g\colon C\longrightarrow A$ be a morphism of filtered coalgebras. Recall that the convolution product $f*g$ is given by the composite

$$C\xrightarrow{\psi}C\otimes C\xrightarrow{f\otimes g}A\otimes A\xrightarrow{\mu}A.$$

Proposition 2.1. [18] *The following statements hold:*

1) *The conjugation $\chi\colon A\longrightarrow A$ is a filtered map for any connected filtered Hopf algebra A.*
2) *For any connected filtered coalgebra C and connected filtered Hopf algebra A, the set $[C,A]$ is a group under the convolution multiplication.*
3) *If C is filtered coalgebra, then E^0C is a bigraded coalgebra. Moreover if A is a filtered Hopf algebra, then E^0A is a bigraded Hopf algebra.*

4) *There are natural isomorphisms of coalgebras*

$$E^0(C \vee D) \cong E^0C \vee E^0D, E^0(C \otimes D) \cong E^0C \otimes E^0D, E^0(C \wedge D) \cong E^0C \wedge E^0D$$

for any pointed filtered coalgebras C and D.

5) *Let C be a pointed filtered coalgebra and let A be a filtered Hopf algebra. Then the function*

$$E^0\colon [C, A] \longrightarrow [E^0C, E^0A], \quad f \longrightarrow E^0 f$$

is a homomorphism of monoids (groups if A is connected). □

2.2. **The Algebraic James Construction.** The James construction $J(C)$ of a pointed filtered coalgebra C is defined as the coequalizer of the diagram with morphisms

$$d^i\colon C^{\otimes n-1} = C \otimes \cdots \otimes C \otimes \overset{(i)}{\mathbf{k}} \otimes C \otimes \cdots \otimes C \longrightarrow C^{\otimes n}$$

for $1 \leq i \leq n < \infty$. The James construction $J(C)$ is the filtered (cocommutative) Hopf algebra which is the coadjoint of the forgetful functor from filtered (cocommutative) Hopf algebras to pointed filtered (cocommutative) coalgebras. The James construction comes with a filtration by tensor length

$$\mathbf{k} = J_0C \subseteq J_1C = C \subseteq J_2C \subseteq \cdots \subseteq J(C)$$

with

$$J(C) = \bigcup_n J_n(C),$$

which we call the *James filtration.* However the filtration on C induces a filtration on $J_n(C)$ for each n, and the James filtration is a filtered filtration. That is, each inclusion $J_{n-1}C \subseteq J_nC$ is a morphism of pointed filtered coalgebras. The resulting sequence

$$J_{n-1}C \longrightarrow J_nC \longrightarrow C^{\wedge n}$$

is a short exact sequence in the category of pointed filtered cocommutative coalgebras.

Proposition 2.2. [18, Proposition 3.6] *Let C be a pointed filtered coalgebra. Then there is a natural isomorphism of bigraded coalgebras*

$$E^0J_n(C) \cong J_n(E^0C)$$

for each $1 \leq n \leq \infty$.

2.3. **The Augmentation Ideal Filtration of $J(C)$.** Let C be a connected coalgebra. The augmentation filtration of C is defined by

$$IC = \mathrm{Ker}(\epsilon\colon C \longrightarrow \mathbf{k})$$

and $I^sC = 0$ for $s \geq 2$. Clearly C is a filtered coalgebra under the augmentation filtration and E^0C is a bigraded coalgebra with trivial comultiplication. Since the filtration on $J(C)$ is induced by the product filtration, the filtration on $J(C)$ is the same as the augmentation ideal filtration of the Hopf algebra $J(C)$. Thus

$$E^0J(C) = J(E^0C) \cong T(IC)$$

is the tensor algebra primitively generated by $E^0_{1,*}J(C) = IC$.

The (product) filtration on $C^{\wedge n}$ is

$$IC^{\wedge n} = I^2C^{\wedge n} = \cdots = I^nC^{\wedge n} = (IC)^{\otimes n}$$

and $I^k C^{\wedge n} = 0$ for $k > n$. Thus in the bigraded coalgebra $E^0 C^{\wedge n}$

$$E^0_{n,*} C^{\wedge n} = (IC)^{\otimes n}$$

is primitive and $E^0_{j,*} C^{\wedge n} = 0$ for $j \neq 0, n$.

2.4. **Group Representations.** In this subsection the ground field is of characteristic p or 0. Let C be a connected filtered coalgebra. Then $J(C)$ is a connected filtered Hopf algebra and so $[C^{\otimes n}, J(C)]$ is a group. For $1 \leq i \leq n$, observe that the composite

$$\phi_i \colon C^{\otimes n} \xrightarrow{\epsilon_{C^{\otimes i-1}} \otimes \mathrm{id}_C \otimes \epsilon_{C^{\otimes n-i}}} C$$

is a filtered coalgebra map and so it induces a group homomorphism

$$\phi_i^* \colon [C, J(C)] \longrightarrow [C^{\otimes n}, J(C)].$$

Lemma 2.3. *Let A be a connected coalgebra and let n be an integer such that $n \not\equiv 0 \mod \mathrm{char}(\mathbf{k})$. Then the convolution power $n \colon A \longrightarrow A$ is an isomorphism. Thus $n^{-1} \colon A \longrightarrow A$ is well-defined.*

Proof. Since $n(x) = nx$ for $x \in PA$, the map $n|_{PA} \colon PA \longrightarrow PA$ is an isomorphism and so $n \colon A \to A$ is a monomorphism. If A is of finite type, then $n \colon A \longrightarrow A$ is an isomorphism. The assertion follows from the fact that [11, Proposition 4.13] any coalgebra A is a direct limit of its sub-Hopf algebras of finite type. □

Let $\mathbb{Z}_{(p)}$ be the ring of p-local integers. By Lemma 2.3, there is a unique group homomorphism

$$e \colon \mathbb{Z}_{(p)} \longrightarrow [C, J(C)]$$

such that $e(1) \colon C \longrightarrow J(C)$ is the canonical inclusion. Let

$$F_n^{\mathbb{Z}_{(p)}} = \coprod_{i=1}^{n} (\mathbb{Z}_{(p)})_{x_i}$$

be the free product, where $(\mathbb{Z}_{(p)})_{x_i}$ is a copy of $\mathbb{Z}_{(p)}$ labeled by letters x_i. Then there is a unique group homomorphism

$$\theta_n \colon F_n^{\mathbb{Z}_{(p)}} \longrightarrow [C^{\otimes n}, J(C)]$$

such that

$$\theta_n|_{(\mathbb{Z}_{(p)})_{x_i}} = \phi_i^* \circ e \colon (\mathbb{Z}_{(p)})_{x_i} = Z_{(p)} \longrightarrow [C^{\otimes n}, J(C)].$$

Write x_i^r for the element $r \in (Z_{(p)})_{x_i} \subseteq F_n^{\mathbb{Z}_{(p)}}$. For $1 \leq i \leq n$, define a face $d_i \colon F_n^{\mathbb{Z}_{(p)}} \longrightarrow F_{n-1}^{\mathbb{Z}_p}$ as the group homomorphism given by

$$d_i x_j^r = \begin{cases} x_j^r & \text{for} \quad j < i, \\ 1 & \text{for} \quad j = i, \\ x_{j-1}^r & \text{for} \quad j > i. \end{cases}$$

Proposition 2.4. *Let C be a connected filtered coalgebra. Then there is a commutative diagram*

$$\begin{array}{ccc} F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [C^{\otimes n}, J(C)] \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i = (d^i)^*} \\ F_{n-1}^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_{n-1}} & [C^{\otimes n-1}, J(C)] \end{array}$$

for $1 \leq i \leq n$*, where*

$$d^i = \mathrm{id}_{C^{i-1}} \otimes \eta_C \otimes \mathrm{id}_{C^{\otimes n-i}} \colon C^{\otimes n-1} = C \otimes \cdots \otimes C \otimes \overset{(i)}{\mathbf{k}} \otimes C \otimes \cdots \otimes C \longrightarrow C^{\otimes n}.$$

Proof. The proof is immediate. □

These definitions can be reformulated in terms of spaces. Let X be a path-connected p-local space. Then there is a unique group homomorphism

$$e\colon \mathbb{Z}_{(p)} \longrightarrow [X, J(X)]$$

such that $e(1)\colon X \longrightarrow J(X)$ is the canonical inclusion. It follows that there is a unique group homomorphism

$$\theta_n \colon F_n^{\mathbb{Z}_{(p)}} \longrightarrow [X^n, J(X)]$$

such that

$$\theta_n|_{(\mathbb{Z}_{(p)})_{x_i}} = \pi_i^* \circ e\colon (\mathbb{Z}_{(p)})_{x_i} = Z_{(p)} \longrightarrow [X^n, J(X)],$$

where $\pi_i\colon X^n \to X$ is the i-coordinate projection. The group homomorphisms from $F_n^{\mathbb{Z}_{(p)}}$ to $[X^n, J(X)]$ and $[C^{\otimes n}, J(C)]$ are both commonly denoted by θ_n since the the homology functor acts as a natural transformation.

Note that for an arbitrary map $f\colon X^n \longrightarrow J(X)$, $f_*\colon H_*(X)^{\otimes n} \longrightarrow J(H_*(X))$ need not preserve the filtration in general. Thus $\mathrm{Im}(\theta_n)$ gives the homotopy classes of a special family of maps $X^n \to J(X)$, which we define as

$$[X^n, J(X)]' = \{[f] \in [X^n, J(X)] \mid f_*\colon H_*(X^n) = H_*(X)^{\otimes n} \longrightarrow J(H_*(X)) \text{ preserves the filtration}\}.$$

Proposition 2.5. *Let X be a path-connected p-local space. Then there is a commutative diagram*

$$\begin{array}{ccc} F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\ \theta_n\ } & [X^n, J(X)]' \\ \| & & \downarrow{\scriptstyle H_*} \\ F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\ \theta_n\ } & [H_*(X)^{\otimes n}, J(H_*(X))]. \end{array}$$

Proof. The proof is immediate. □

Let

$$\mathfrak{H}F_n^{\mathbb{Z}_{(p)}} = \{w \in F_n^{\mathbb{Z}_{(p)}} \mid d_1 w = d_2 w = \cdots = d_n w\}$$

be the equalizer of the faces d_i for $1 \leq i \leq n$. Note that

$$d_1(\mathfrak{H}F_n^{\mathbb{Z}_{(p)}}) \subseteq \mathfrak{H}F_{n-1}^{\mathbb{Z}_{(p)}}.$$

Let $p_n = d_1|_{\mathfrak{H}F_n^{\mathbb{Z}_{(p)}}} \colon \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} \longrightarrow \mathfrak{H}F_{n-1}^{\mathbb{Z}_{(p)}}$. According to [22, Proposition 1.2.1], each

$$p_n \colon \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} \longrightarrow \mathfrak{H}F_{n-1}^{\mathbb{Z}_{(p)}}$$

is an epimorphism. Let $q_n \colon C^{\otimes n} \longrightarrow J_n(C)$ be the quotient map and let $i_{n-1}\colon J_{n-1}(C) \longrightarrow J_n(C)$ be the inclusion map.

Proposition 2.6. *There is a commutative diagram*

$$\begin{array}{ccc} F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [C^{\otimes n}, J(C)] \\ \uparrow & & \uparrow q_n^* \\ \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [J_n(C), J(C)] \end{array}$$

for each n. Moreover there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [J_n(C), J(C)] \\ \downarrow p_n & & \downarrow i_n^* \\ \mathfrak{H}F_{n-1}^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_{n-1}} & [J_{n-1}(C), J(C)] \end{array}$$

for each n.

Proof. The first diagram follows from the fact that $J_n(C)$ is the coequalizer of the morphisms

$$d^i \colon C^{\otimes n-1} = C \otimes \cdots \otimes C \otimes \overset{(i)}{\mathbf{k}} \otimes C \otimes \cdots \otimes C \longrightarrow C^{\otimes n}$$

for $1 \leq i \leq n$. The second diagram follows from the first together with Proposition 2.4 in the $i = 1$ case. □

Corollary 2.7. *For any connected filtered coalgebra C, there is a representation*

$$\theta^C \colon \mathfrak{H}F^{\mathbb{Z}_{(p)}} = \lim_{\overleftarrow{p_n}} \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} \longrightarrow [J(C), J(C)] = \lim_{\overleftarrow{i_{n-1}^*}} [J_n(C), J(C)]$$

of the progroup $\mathfrak{H}F^{\mathbb{Z}_{(p)}}$.

Let $K_n^{\mathbb{Z}_{(p)}}$ be the quotient group of $F_n^{\mathbb{Z}_{(p)}}$ subject to the following iterated commutator relations

$$[[[x_{i_1}^{r_1}, x_{i_2}^{r_2}], x_{i_3}^{r_3}], \ldots, x_{i_t}^{r_t}] = 1 \tag{2.1}$$

if $i_a = i_b$ for some $1 \leq a < b \leq t$.

Lemma 2.8. *For each $1 \leq i \leq n$, the composite*

$$F_n^{\mathbb{Z}_{(p)}} \xrightarrow{d_i} F_{n-1}^{\mathbb{Z}_{(p)}} \longrightarrow K_{n-1}^{\mathbb{Z}_{(p)}}$$

factors through the quotient $K_n^{\mathbb{Z}_{(p)}}$ and so there is a (unique) homomorphism

$$d_i \colon K_n^{\mathbb{Z}_{(p)}} \longrightarrow K_{n-1}^{\mathbb{Z}_{(p)}}$$

such that the diagram

$$\begin{array}{ccc} F_n^{\mathbb{Z}_{(p)}} & \twoheadrightarrow & K_n^{\mathbb{Z}_{(p)}} \\ \downarrow d_i & & \downarrow d_i \\ F_{n-1}^{\mathbb{Z}_{(p)}} & \twoheadrightarrow & K_{n-1}^{\mathbb{Z}_{(p)}} \end{array}$$

commutes.

Proof. The proof is straightforward. □

Lemma 2.9. *Let C be a connected coalgebra filtered by*

$$IC = \mathrm{Ker}(\epsilon\colon C \longrightarrow \mathbf{k})$$

and $I^tC = 0$ for $t > 1$. Then there is a commutative diagram

$$\begin{array}{ccc} F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [C^{\otimes n}, J(C)] \\ \downarrow & & \downarrow E^0 \\ K_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [E^0C^{\otimes n}, E^0J(C)] \end{array}$$

Proof. By Propositions 2.1 and 2.2, for $x_i^r \in F_n^{\mathbb{Z}_{(p)}}$, the map $E^0(\theta_n(x_i^r))$ is the composite

$$E^0C^{\otimes n} = (E^0C)^{\otimes n} \xrightarrow{\mathrm{proj_i}} E^0C \xrightarrow{r} E^0C \hookrightarrow E^0J(C) = J(E^0C),$$

where $r\colon E^0C \longrightarrow E^0C$ is the coalgebra map such that

$$r\colon E^0_{1,*}C = IC \longrightarrow E^0_{1,*}C = IC$$

is multiplication by r. (*Note:* $E^0_{0,*}C = \mathbf{k}$, $E^0_{1,*}C = IC$ and $E^0_{t,*}C = 0$ for $t > 1$.)

Let

$$w = [[[x_{i_1}^{r_1}, x_{i_2}^{r_2}], x_{i_3}^{r_3}], \ldots, x_{i_t}^{r_t}] \in F_n^{\mathbb{Z}_{(p)}}$$

such that $i_a = i_b$ for some $1 \le a < b \le t$. Then $E^0(\theta_n(w))$ is the composite

$$E^0C^{\otimes n} = (E^0C)^{\otimes n} \xrightarrow{\pi_{i_1,i_2,\ldots,\hat{i_b},\ldots,i_t}} (E^0C)^{\otimes t-1} \twoheadrightarrow (E^0C)^{\wedge t-1}$$

$$\xrightarrow{\mathrm{id}_{C^{a-1}}\wedge\bar{\psi}\wedge\mathrm{id}_{C^{t-a-1}}} (E^0C)^{\wedge t} \xrightarrow{T_{a+1,b}} (E^0C)^{\wedge t} \xrightarrow{r_1\wedge\cdots\wedge r_t} (E^0C)^{\wedge t} \xrightarrow{S_t} J(E^0C),$$

where $\pi_{i_1,i_2,\ldots,\hat{i_b},\ldots,i_t}$ is the coordinate projection map, $\bar{\psi}$ is the reduced comultiplication, $T_{a+1,b}$ is the map switching positions $a+1$ and b and S_t is the algebraic Samelson map defined in [18]. Since the reduced comultiplication

$$\bar{\psi}\colon C \longrightarrow C \wedge C$$

is the trivial map, $E^0(\theta_n(w)) = 1$ and hence the result. □

Using the face maps in Lemma 2.9, let

$$\mathfrak{H}K_n^{\mathbb{Z}_{(p)}} = \{w \in K_n^{\mathbb{Z}_{(p)}} \mid d_1w = d_2w = \cdots = d_nw\}$$

be the equalizer of the faces d_i for $1 \le i \le n$ and let $\mathfrak{H}K^{\mathbb{Z}_{(p)}} = \varprojlim_{p_n} \mathfrak{H}K_n^{\mathbb{Z}_{(p)}}$, where $p_n = d_1|_{\mathfrak{H}K_n^{\mathbb{Z}_{(p)}}}$.

Theorem 2.10. *Let C be any connected coalgebra.*

1) *For any word $w \in \mathfrak{H}F^{\mathbb{Z}_{(p)}}$, the coalgebra map*

$$\theta^C(w)\colon J(C) \longrightarrow J(C)$$

preserves the augmentation ideal filtration.

2) *Let $J(C)$ be filtered by the augmentation ideal. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathfrak{H}F^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta^C} & [J(C), J(C)] \\ \downarrow & & \downarrow E^0 \\ \mathfrak{H}K^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta} & [E^0J(C), E^0J(C)]. \end{array}$$

Proof. Let C be filtered by $IC = \mathrm{Ker}(\epsilon\colon C \longrightarrow \mathbf{k})$ and $I^tC = 0$ for $t > 1$. According to Subsection 2.3, the filtration on $J(C)$ induced from C is the same as the augmentation ideal filtration.

(1). For any word $w \in \mathfrak{H}F^{\mathbb{Z}_{(p)}}$, by Corollary 2.7,

$$\theta^C(w)\colon J(C) \longrightarrow J(C)$$

is a filtered coalgebra map with respect to the filtration of $J(C)$ induced from C. Thus $\theta^C(w)\colon J(C) \longrightarrow J(C)$ preserves the augmentation ideal filtration, which is the assertion.

(2). By Lemma 2.9, there is a commutative diagram

$$\begin{array}{ccc} F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [C^{\otimes n}, J(C)] \\ \downarrow & & \downarrow E^0 \\ K_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [E^0C^{\otimes n}, E^0J(C)]. \end{array}$$

By taking equalizers, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [J_n(C), J(C)] \\ \downarrow & & \downarrow E^0 \\ \mathfrak{H}K_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [E^0J_n(C), E^0J(C)]. \end{array}$$

The assertion follows by taking inverse limits. □

Reformulating in terms of spaces again, let X be a path-connected p-local space. By [21, Lemma 2.9], for any Y, $[J_n(X), \Omega Y]$ is the equalizer of the maps

$$d^{i*}\colon [X^n, \Omega Y] \longrightarrow [X^{n-1}, \Omega Y]$$

for $1 \le i \le n$, where $d^i\colon X^{n-1} \longrightarrow X^n$ with

$$d^i(z_1, z_2, \ldots, z_{n-1}) = (z_1, \ldots, z_{i-1}, *, z_i, \ldots, z_{n-1}).$$

Let

$$[J_n(X), J(X)]' = \{[f] \in [J_n(X), J(X)] \mid f_*\colon J_n(H_*(X)) \longrightarrow J(H_*(X)) \text{ preserves the filtration}\}.$$

Just as in Proposition 2.5, applying the homology functor gives:

Proposition 2.11. *Let X be a path-connected p-local space. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [J_n(X), J(X)]' \\ \Vert & & \downarrow H_* \\ \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta_n} & [J_n(H_*(X)), J(H_*(X))] \end{array}$$

Proof. The proof is immediate. □

Taking $Y = \Omega\Sigma X$, the analogue of Proposition 2.6 for spaces gives a commutative diagram

$$\begin{array}{ccccc}
F_n^{\mathbb{Z}_{(p)}} & \hookleftarrow & \mathfrak{H}F_n^{\mathbb{Z}_{(p)}} & \xrightarrow{p_n} & \mathfrak{H}F_{n-1}^{\mathbb{Z}_{(p)}} \\
\downarrow{\scriptstyle\theta_n} & & \downarrow{\scriptstyle\theta_n} & & \downarrow{\scriptstyle\theta_{n-1}} \\
[X^n, J(X)] & \hookleftarrow & [J_n(X), J(X)] & \twoheadrightarrow & [J_{n-1}(X), J(X)].
\end{array}$$

Let

$$[J(X), J(X)]' = \{[f] \in [J(X), J(X)] \mid f_*\colon J(H_*(X)) \longrightarrow J(H_*(X)) \text{ preserves the filtration}\}.$$

Corollary 2.12. *Let X be a path-connected p-local space and let*

$$H_*(J(X)) = J(H_*(X))$$

be filtered by augmentation ideal. Then there is a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{H}F^{\mathbb{Z}_{(p)}} & = & \mathfrak{H}F^{\mathbb{Z}_{(p)}} & \twoheadrightarrow & \mathfrak{H}K^{\mathbb{Z}_{(p)}} \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\
[J(X), J(X)]' & \xrightarrow{H_*} & [J(H_*(X)), J(H_*(X))] & \xrightarrow{E^0} & [E^0J(H_*(X)), E^0J(H_*(X))].
\end{array}$$

Proof. In the diagram immediately preceeding the Corollary, apply the homology functor, use Proposition 2.11, and take inverse limits to obtain the commutativity of the left square in the asserted diagram. The right square is an application of Theorem 2.10 (2). □

2.5. **The Progroup $\mathfrak{H}K^{\mathbb{Z}_{(p)}}$.** Let $\mathcal{M}_R$ be the category of projective R-modules. For each V in $\mathcal{M}_R$, $T(V)$ is a Hopf algebra by saying V is primitive. Thus T is a functor from $\mathcal{M}_R$ to the category of Hopf algebras. By forgetting the algebraic structure, T is a functor from $\mathcal{M}_R$ to the category of connected coalgebras. Let $\mathrm{coalg}^R(T, T)$ denote the group of self coalgebra natural transformations of T. The proof of the following theorem follows along the lines of [17, Sections 2-3].

Theorem 2.13. *The progroup $\mathfrak{H}K^{\mathbb{Z}_{(p)}}$ has the following properties:*

1) *There is an isomorphism of progroups*

$$\Phi\colon \mathfrak{H}K^{\mathbb{Z}_{(p)}} \xrightarrow{\cong} \mathrm{coalg}^{\mathbb{Z}_{(p)}}(T, T).$$

2) *Let the ground field $\mathbf{k}$ be of characteristic 0 or p and let C be any connected coalgebra. Let $J(C)$ be filtered by the augmentation ideal. There is a commutative diagram of progroups*

$$\begin{array}{ccccc}
\mathfrak{H}K^{\mathbb{Z}_{(p)}} & \xrightarrow{\theta} & [E^0J(C), E^0J(C)] & = & [T(IC), T(IC)] \\
{\scriptstyle\cong}\downarrow{\scriptstyle\Phi} & & & & \uparrow{\scriptstyle\text{evaluation}} \\
\mathrm{coalg}^{\mathbb{Z}_{(p)}}(T, T) & & \longrightarrow & & \mathrm{coalg}^{\mathbf{k}}(T, T).
\end{array}$$

□

Combining Corollary 2.12 and Theorem 2.13 when $\mathbf{k} = \mathbb{Z}/p$ gives the following.

Theorem 2.14. *There is a commutative diagram*

$$\begin{array}{ccccc} \mathfrak{H}F^{\mathbb{Z}_{(p)}} & \twoheadrightarrow & \mathfrak{H}K^{\mathbb{Z}/p} & \xrightarrow{\cong} & \mathrm{coalg}^{\mathbb{Z}_{(p)}}(T,T) \\ \downarrow{\scriptstyle\theta} & & & & \downarrow \\ [J(X),J(X)]' & & \xrightarrow{E^0H_*(-;\mathbb{Z}/p)} & & \mathrm{coalg}^{\mathbb{Z}/p}(T,T) \end{array}$$

□

Finally, we relate the idempotents in $\mathrm{coalg}^R(T,T)$ for various rings R. The proof of the following was given in [18, Section 2].

Theorem 2.15. *Let* $\mathbf{k}$ *be a field of characteristic* p. *The maps given by changing ground rings*

$$\phi_{\mathbb{Z}_{(p)},\mathbb{Z}/p} \colon \mathrm{coalg}^{\mathbb{Z}_{(p)}}(T,T) \longrightarrow \mathrm{coalg}^{\mathbb{Z}/p}(T,T)$$
$$\phi_{\mathbb{Z}/p,\mathbf{k}} \colon \mathrm{coalg}^{\mathbb{Z}/p}(T,T) \longrightarrow \mathrm{coalg}^{\mathbf{k}}(T,T)$$

induce bijections between idempotent elements. □

3. Proof of Theorems 1.1 and 1.2

Let Y be any p-local simply-connected co-H space. So Y has a co-H structure map $s\colon Y \longrightarrow Y\vee Y$ which is a right homotopy inverse to the evaluation map $\sigma\colon \Sigma\Omega Y \longrightarrow Y$. We require a lemma on how $\Omega\sigma$ acts in homology.

Lemma 3.1. *Let* Y *be a* p*-local simply-connected co-*H *space. Then there is a commutative diagram of Hopf algebras*

$$\begin{array}{ccc} E^0H_*(J(\Omega Y)) & \xrightarrow{E^0r} & E^0H_*(\Omega Y) \\ \| & & \downarrow{\scriptstyle\cong} \\ T(\bar{H}_*(\Omega Y)) & \xrightarrow{T(q)} & T(\Sigma^{-1}\bar{H}_*(Y)) \end{array}$$

where $q\colon \bar{H}_*(\Omega Y) \longrightarrow Q(H_*(\Omega Y)) = \Sigma^{-1}\bar{H}_*(Y)$ *is the canonical quotient.*

Proof. Observe that $E^0H_*(J(\Omega Y))$ is the tensor algebra primitively generated by $E^0_{1,*}H_*(J(\Omega Y)) = IH_*(\Omega Y)$. Thus

$$E^0\Omega r\colon E^0J(H_*(\Omega Y)) = T(IH_*(\Omega Y)) \longrightarrow E^0H_*(\Omega Y)$$

is the (unique) morphism of Hopf algebras induced by the composite

$$IH_*(\Omega Y) \xrightarrow{\mathrm{proj}} Q(H_*(\Omega Y)) = E^0_{1,*}(H_*(\Omega Y)) \hookrightarrow E^0H_*(\Omega Y).$$

The lemma now follows. □

Proof of Theorem 1.1: Let the ground field $\mathbf{k} = \mathbb{Z}/p$ and $f_V\colon T(V) \longrightarrow T(V)$ be any functorial coalgebra idempotent. By Theorems 2.14 and 2.15, there is a word $w_f \in \mathfrak{H}F^{\mathbb{Z}_{(p)}}$ such that the image of w_f in $\mathfrak{H}K^{\mathbb{Z}_{(p)}} \cong \mathrm{coalg}^{\mathbb{Z}_{(p)}}(T,T)$ is an idempotent which reduces to $[f_V]$ modulo p. Further, using the representation

$$\theta\colon \mathfrak{H}F^{\mathbb{Z}_{(p)}} \longrightarrow [J(\Omega Y), J(\Omega Y)]',$$

there is a map $\theta(w_f)\colon J(\Omega Y) \longrightarrow J(\Omega Y)$ such that in mod-p homology

$$\theta(w_f)_*\colon H_*(J(\Omega Y)) \longrightarrow H_*(J(\Omega Y))$$

is an idempotent which preserves the augmentation ideal filtration.

Define $e(w_f)$ by the composite

$$e(w_f)\colon \Omega Y \xrightarrow{\Omega s} J(\Omega Y) \xrightarrow{\theta(w_f)} J(\Omega Y) \xrightarrow{\Omega\sigma} \Omega Y. \tag{3.1}$$

By Lemma 3.1, $H_*(J(\Omega Y)) \cong T(H_*(\Omega Y)) \overset{(\Omega r)_*}{\longrightarrow} H_*(\Omega Y) \cong T(\Sigma^{-1}\bar{H}_*(Y))$ is determined by projecting onto the generating set. The functoriality of f_V therefore implies the right square in the following diagram commutes

$$\begin{array}{ccccc} E^0H_*(\Omega Y) & \xrightarrow{E^0(\Omega s)_*} & T(\bar{H}_*(\Omega Y)) & \xrightarrow{\theta(w_f)_*=[f_V]} & T(\bar{H}_*(\Omega Y)) \\ \| & & \downarrow{\scriptstyle (\Omega r)_*} & & \downarrow{\scriptstyle (\Omega r)_*} \\ E^0H_*(\Omega Y) & = & T(\Sigma^{-1}\bar{H}_*(Y)) & \xrightarrow{[f_{\Sigma^{-1}\bar{H}_*(Y)}]} & T(\Sigma^{-1}\bar{H}_*(Y)). \end{array}$$

The left square commutes because $s\circ r$ is homotopic to the identity map on Y. Note that the upper direction around the diagram is $e(w_f)_*$. The lower direction around the diagram is an idempotent because $[f_{\Sigma^{-1}\bar{H}_*(Y)}]$ is. Thus $e(w_f)_*$ is an idempotent. Hence there is a functorial decomposition

$$\Omega Y \simeq \mathrm{hocolim}_{e(w_f)}\Omega Y \times \mathrm{hocolim}_{1-e(w_f)}\Omega Y$$

with $E^0H_*(\mathrm{hocolim}_{e(w_f)}\Omega Y) \cong f_{\Sigma^{-1}\bar{H}_*(Y)}(T(\Sigma^{-1}\bar{H}_*(Y)))$. This proves the theorem. □

Proof of Theorem 1.2: In Theorem 1.1, choose the idempotents $f_V \in \mathrm{coalg}^{\mathbb{Z}/p}(T,T)$ corresponding to the functors $A^{\min}$ and $B^{\max}$. □

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Department of Mathematics, University of Toronto, Toronto, Ontario M5S2E4, Canada
E-mail address: selick@math.toronto.edu
URL: http://www.math.toronto.edu/selick

Department of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, United Kingdom
E-mail address: s.theriault@maths.abdn.ac.uk
URL: http://maths.abdn.ac.uk/~stephen

Department of Mathematics, National University of Singapore, Singapore 117543, Singapore
E-mail address: matwuj@nus.edu.sg
URL: http://www.math.nus.edu.sg/~matwujie