# Introduction to Algebraic Topology

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# Chapter 1

# Introduction

## 1.1 Sets

Let $X$ and $Y$ be sets. The notation $Y \subseteq X$ means that $Y$ is a subset of $X$ and $Y \subset X$ means that $Y$ is a proper subset of $X$, that is $Y \subseteq X$ and $Y \neq X$. Let $X \setminus Y$ denote the set

$$X \setminus Y = \{x | x \in X \quad \text{and} \quad x \notin Y\}.$$

The empty set is denoted by $\emptyset$.

Let $X$ and $Y$ be sets. The Cartesian product is defined by

$$X \times Y = \{(x, y) | x \in X, y \in Y\}.$$

**Note:** If $X$ and $Y$ are finite sets of $m$ and $n$ elements, respectively, then $X \times Y$ is a finite set of $mn$ elements.

Let $X_1, \cdots, X_n$ be sets. The Cartesian product is defined by

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \cdots, x_n) | x_i \in X_i, 1 \leq i \leq n\}.$$

The infinite Cartesian product is defined similarly. For example, let $\{X_\alpha | \alpha \in I\}$ be a family of sets. Then

$$\prod_{\alpha \in I} X_\alpha = \{(x_\alpha) | x_\alpha \in X_\alpha, \alpha \in I\}.$$

The $\alpha$-coordinate projection

$$\pi_\alpha \colon \prod_{\alpha' \in I} X_{\alpha'} \to X_\alpha$$

is defined by

$$\pi_\alpha((x_{\alpha'})) = x_\alpha.$$

**Theorem 1.1.1** *Let $\{X_\alpha | \alpha \in I\}$ be a family of set. Then the Cartesian product $\prod_{\alpha \in I} X_\alpha$ satisfies the following universal lifting property:*

> *Let $X$ be any set and let $f_\alpha \colon X \to X_\alpha$ be any function for each $\alpha \in I$. Then there is a unique function*
>
> $$f \colon X \to \prod_{\alpha \in I} X_\alpha$$
>
> *such that*
>
> $$f_\alpha = \pi_\alpha \circ f$$
>
> *for each $\alpha$.*

*Proof.* Let $f \colon X \to \prod_{\alpha \in I} X_\alpha$ be a function defined by

$$f(x) = (f_\alpha(x))$$

for each $x \in X$. Then $f$ is a function with the property that $f_\alpha(x) = \pi_\alpha \circ f(x)$ for any $x$ and so $f_\alpha = \pi_\alpha \circ f$. This shows the existence of the universal lifting property. Let $g \colon X \to \prod_{\alpha \in I} X_\alpha$ be any function with the property that $f_\alpha = \pi_\alpha \circ g$ for each $\alpha$. Then the $\alpha$-th coordinate of $g(x)$ is $f_\alpha(x)$ for each $x \in X$. Thus $g = f$ defined above. This shows the uniqueness of the universal lifting property.

Let $f \colon X \to Y$ be a function. Then the image of $f$ is defined by

$$\mathrm{Im}(f) = f(X) = \{y \in Y | y = f(x) \quad \text{for} \quad \text{some} \quad x \in X\}.$$

The identity function on $X$ is denoted by $\mathrm{id}_X$, id or 1. Thus $\mathrm{id}(x) = x$.

**Exercise 1.1.1** *Let $f \colon X \to Y$ be a function. Let $\{X_\alpha | \alpha \in I\}$ be a family of subsets of $X$. Then*

*1) show that*

$$f(\bigcup_{\alpha \in I} X_\alpha) = \bigcup_{\alpha \in I} f(X_\alpha);$$

*2) show that*

$$f(\bigcap_{\alpha \in I} X_\alpha) \subseteq \bigcap_{\alpha \in I} f(X_\alpha);$$

*3) show by example that*

$$f(\bigcap_{\alpha\in I} X_\alpha) \neq \bigcap_{\alpha\in I} f(X_\alpha)$$

*in general.*

Let $f\colon X \to Y$ be a function. Let $A$ be a subset of $X$. Then the restriction $f|_A\colon A \to Y$ is the function defined by

$$f|_A(a) = f(a)$$

for $a \in A$. Let $B$ be a subset of $Y$. The pre-image $f^{-1}(B)$ is defined by

$$f^{-1}(B) = \{x \in X | f(x) \in B\}.$$

Note that $f^{-1}(B)$ could be an empty set.

**Exercise 1.1.2** *Let $f\colon X \to Y$ be a function. Let $\{B_\beta | \beta \in J\}$ be a family of subsets of $Y$. Then*

*1) show that*

$$f^{-1}(\bigcup_{\beta\in J} B_\beta) = \bigcup_{\beta\in J} f^{-1}(B_\beta);$$

*2) show that*

$$f^{-1}(\bigcap_{\beta\in J} B_\beta) = \bigcap_{\beta\in J} f^{-1}(B_\beta);$$

*3) show that*

$$f^{-1}(Y \setminus B_\beta) = X \setminus f^{-1}(B_\beta).$$

A function $f\colon X \to Y$ is said to be bijective if it is one-to-one and onto. In this case the inverse is denoted by $f^{-1}\colon Y \to X$. Note that $f^{-1}$ is also bijective. If there is a bijective function $f$ from $X$ to $Y$, we call that $X$ is isomorphic to $Y$ as sets.

**Exercise 1.1.3** *Let $X$ be a set. Let $X_\alpha$ be a family of sets with indices $\alpha$ in a set $I$. Suppose that $X_\alpha = X$ for each $\alpha$. Show that $\prod_{\alpha\in I} X_\alpha$ is isomorphic to the set of functions from $I$ to $X$.*

A relation on a set $X$ is a subset $\sim$ of $X \times X$. We write $x \sim y$ if $(x, y) \in \sim$. A relation on $X$ is an *equivalence relation* if it satisfies

1) the reflexive condition: $x \sim x$ for all $x \in X$;

2) the symmetric condition: If $x \sim y$, then $y \sim x$;

3) the transitive condition: If $x \sim y$ and $y \sim z$, then $x \sim z$.

The equivalence class of $x$ is the set

$$\{x\} = \{y \in X | x \sim y\}.$$

**Exercise 1.1.4** *Let $\sim$ be an equivalence relation on $X$. Show that each element of $X$ belongs to exactly one equivalence class.*

## 1.2 Monoids and Groups

A *binary operation (multiplication)* on a set $X$ is a function $\mu: X \times X \to X$. We abbreviate $\mu(x, y)$ to $xy$ or $x+y$. A *monoid* $M$ is a set $M$ together with a multiplication $\mu: M \times M \to M$ satisfying the following conditions:

1) (identity) there exists an element $1 \in M$ such that

$$1x = x1 = x$$

for any $x \in M$;

2) (associativity) the equation

$$(x_1 x_2)x_3 = x_1(x_2 x_3)$$

holds for any $x_1, x_2, x_3 \in M$.

A *group* is a monoid $G$ satisfying

3) (inverse) For each $x \in G$, there exists an element $x^{-1} \in G$ such that

$$xx^{-1} = x^{-1}x = 1.$$

In other words, a group is a monoid in which every element is invertible. Note that if $x$ is invertible, then the inverse of $x$ is unique. A group (or monoid) $G$ is said to *abelian* or *commutative* if $xy = yx$ for any $x, y \in G$. Let $G$ and $H$ be monoids (or groups). Then the Cartesian product $G \times H$ is a monoid (or group) under the multiplication defined by

$$(g,h)(g',h') = (gg',hh').$$

In additive case, we write $G \oplus H$ for $G \times H$.

A subset $H$ of a group (monoid) is a *subgroup (submonoid)* of $G$ if $H$ is a group (monoid) under the binary operation of $G$. Let $H$ be a subgroup (submonoid) of $G$ and let $g \in G$. The left and right cosets of $H$ by $g$ are defined by

$$gH = \{gh | h \in H\} \qquad Hg = \{hg | h \in H\}.$$

**Example 1.2.1** Let $\mathbb{Z}^+$ be the set of non-negative integers. Then $\mathbb{Z}^+$ is a monoid under the addition $+$. $\mathbb{Z}^+$ is a submonoid of $\mathbb{Z}$. $\mathbb{Z}$ is often called the group completion of the monoid $\mathbb{Z}^+$, i.e. the "smallest group" that contains $\mathbb{Z}^+$. The set of natural numbers is a monoid under the multiplication. The "group completion" of natural numbers is the set of positive rational numbers with the multiplication.

In general, monoids and the "group completion" of monoids are very complicated and there are many research papers about these topics.

Let $G$ and $H$ be monoids (or groups). A homomorphism $f\colon G \to H$ is a function such that $f(1) = 1$ and

$$f(xy) = f(x)f(y)$$

for any $x, y \in G$.

**Exercise 1.2.1** *Let $G$ and $H$be groups and let $f\colon G \to H$ be a function such that $f(xy) = f(x)f(y)$ for any $x, y \in G$. Show that*

*1) $f(1) = 1$;*

*2) $f(x^{-1}) = (f(x))^{-1}$ for any $x \in G$.*

Let $G$ and $H$ be monoids (or groups). The *kernel* of a homomorphism $f\colon G \to H$ is the set

$$\mathrm{Ker}(f) = \{x \in G | f(x) = 1\}.$$

Note that a homomorphism $f$ is one-to-one (a monomorphism) if and only if

$$\mathrm{Ker}(f) = \{1\}.$$

A monoid (or group) $G$ is called isomorphic to $H$ if there is a bijective homomorphism $f\colon G \to H$. In this case, we write $G \cong H$ or $f\colon G \cong H$.

A subgroup $K$ of $G$ is normal if $gxg^{-1} \in K$ for all $g \in G$ and $x \in K$. Let $G$ and $H$ be groups. Then the kernel of a homomorphism $f\colon G \to H$ is a normal subgroup of $G$. The image of $f$ is a subgroup of $H$ which is not normal in general.

**Exercise 1.2.2** *Let $K$ be a normal subgroup of a group $G$. Show that*

*1) $gK = Kg$ for any $g \in G$;*

*2) the set*

$$G/K = \{gK | g \in G\}$$

*is a group under the operation*

$$(gK)(g'K) = (gg')K.$$

*The group $G/K$ is called the quotient group of $G$ by $K$.*

Let $G$ be a group and let $g \in G$. The subgroup generated by $g$ is the subset

$$\langle g \rangle = \{g^n | n \in \mathbb{Z}\}.$$

**Proposition 1.2.2** *Let $G$ be a group and let $g \in G$. Then $\langle g \rangle$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some $n$.*

*Proof.* Let $\phi: \mathbb{Z} \to \langle g \rangle$ be the function defined by

$$\phi(n) = g^n.$$

Then $\phi$ is a homomorphism of groups because

$$\phi(m+n) = g^{m+n} = g^m g^n = \phi(m)\phi(n).$$

Note that $\phi$ is an epimorphism, that is $\phi$ is onto. If $g^m \neq 1$ for any positive integer $m$, then $\phi$ is an isomorphism. Suppose that $g^m = 1$ for some positive integer $m$. Let

$$n = \min\{m | g^m = 1, m > 0\}.$$

Then

$$\mathrm{Ker}(\phi) = n\mathbb{Z}$$

and so

$$\langle g \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$

If $G = \langle g \rangle$ for some $g$, we say that $G$ is a *cyclic* group with generator $g$. A set of *generators* for a group $G$ is a subset $S$ of $G$ such that each element in $G$ is a product of powers of elements taken from $S$. A group $G$ is called *finitely generated* if it is generated by a finite subset.

A *free abelian group* of rank $n$ is the direct sum

$$\mathbb{Z}^{\oplus n} = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.$$

**Theorem 1.2.3 (Decomposition Theorem)** *Let $G$ be a finitely generated abelian group. Then $G$ is isomorphic to*

$$H_0 \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_m,$$

*where $H_0$ is a free abelian group and $H_i$ is a cyclic group of prime power order for $1 \leq i \leq m$.*

The proof can be found in any text book of algebra.

A commutator in a group $G$ is an element

$$[g, h] = ghg^{-1}h^{-1}.$$

for some elements $g, h \in G$. The commutator subgroup $[G, G]$ is the subgroup of $G$ generated by all commutators of $G$. The commutator subgroup $[G, G]$ is normal. The group $G/[G, G]$ is called the *abelianization* of the group $G$. Note that a group $G$ is abelian if and only if the commutator subgroup $[G, G]$ is trivial. A group $G$ is called perfect if $[G, G] = G$. An example of perfect groups is the alternating groups $A_n$ for $n > 4$. Non-commutative groups are much more complicated than abelian groups.

## 1.3 $G$-sets

Let $G$ be a group. A set $X$ is called a *left $G$-set* if there is an operation $\mu: G \times X \to X$, $(g, x) \to g \cdot x$, such that

1) $1 \cdot x = x$ for all $x \in X$;

2) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

A set $X$ is called a *right $G$-set* if there is an operation $\mu: X \times G \to X$, $(g, x) \to x \cdot g$, such that

1) $x \cdot 1 = x$ for all $x \in X$;

2) $x \cdot (gh) = (x \cdot g) \cdot h$ for all $g, h \in G$ and $x \in X$.

**Example 1.3.1** Let $H$ be a subgroup of a group $G$. Then the set of left cosets $\{gH | g \in G\}$ is a left $G$-set and the set of right cosets $\{Hg | g \in G\}$ is a right $G$-set.

**Theorem 1.3.2** *Let $X$ be a left $G$-set. For any $g \in G$, the function $\theta_g: X \to X$ defined by*

$$x \to g \cdot x$$

*is a bijective.*

*Proof.* From the definition, we have that $\theta_g \theta_h = \theta_{gh}$ and $\theta_1 = \mathrm{id}_X$. Thus

$$\theta_g \theta_{g^{-1}} = \mathrm{id}_X = \theta_{g^{-1}} \theta_g$$

and so $\theta_g$ is a bijective.

Similarly, if $X$ is a right $G$-set, then the function $\theta_g: X \to X$ defined by $x \to x \cdot g$ is a bijective.

## 1.4 Categories and Functors

A category may be thought of intuitively as consisting of sets, possibly with additional structure, and functions, possibly preserving additional structure. More precisely, a category $\mathcal{C}$ consists of

1) A class of objects

2) For every ordered pair of objects $X$ and $Y$, a set $\mathrm{Hom}(X, Y)$ of *morphisms* with *domain* $X$ and *range* $Y$; if $f \in \mathrm{Hom}(X, Y)$, we write $f: X \to Y$ or $X \xrightarrow{f} Y$

3) For every ordered triple of objects $X, Y$ and $Z$, a function associating to a pair of morphisms $f: X \to Y$ and $g: Y \to Z$ their *composite*

$$g \circ f: X \to Z$$

These satisfy the following two axioms:

*Associativity.* If $f: X \to Y$, $g: Y \to Z$ and $h: Z \to W$, then

$$h \circ (g \circ f) = (h \circ g) \circ f: X \to W.$$

*Identity.* For every object $Y$ there is a morphism $\mathrm{id}_Y: Y \to Y$ such that if $f: X \to Y$, then $\mathrm{id}_Y \circ f = f$, and if $h: Y \to Z$, then $h \circ \mathrm{id}_Y = h$.

A category is said to be *small* if the class of objects is a set. The category of sets means the category in which the objects are sets and the morphisms are functions. The category of sets is NOT small. But there are many small categories. For instance, the category of finite sets, that is in which the objects are finite sets and the morphisms are functions between finite sets. We list some examples of categories:

1) The category of sets and functions.

2) The category of pointed sets (A pointed set means a non-empty set $X$ with a *base point* $x_0 \in X$) and pointed functions (that is the functions that preserving the base points).

3) The category of finite ordered sets and monotone functions (that is $f(x) \leq f(y)$ is $x \leq y$). This category is usually denoted by $\Delta$. The objects in $\Delta$ are given by $\{0, 1, \cdots, n\}$ for $n \geq 0$ and the morphisms in $\Delta$ are given by monotone function from $\{0, 1, \cdots, m\}$ to $\{0, 1, \cdots, n\}$ for any $m, n$.

4) The category of groups and homomorphisms.

5) The category of monoids and homomorphisms.

6) The category of topological spaces and continuous functions. Topological space is a generalization of the usual spaces such as Euclidian spaces $\mathbb{R}^n$, spheres, *polyhedra*, metric spaces and etc. We will give the definition of topological space in the next chapter.

Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{C}' \subseteq \mathcal{C}$ is a category such that

a) The objects of $\mathcal{C}'$ are also objects of $\mathcal{C}$;

b) For objects $X'$ and $Y'$ of $\mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X', Y')$ is a subset of $\mathrm{Hom}_{\mathcal{C}}(X', Y')$ and

c) If $f' \colon X' \to Y'$ and $g' \colon Y' \to Z'$ are morphisms of $\mathcal{C}'$, their composite in $\mathcal{C}'$ equals their composite in $\mathcal{C}$.

$\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and for objects $X'$ and $Y'$ in $\mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X', Y') = \mathrm{Hom}_{\mathcal{C}}(X', Y')$. For example, the category of groups and homomorphisms is a subcategory of the category of sets and functions but it is not a full subcategory. The category of finite sets and functions is a full subcategory of the category of sets and functions.

Let $\mathcal{C}$ be a category. A morphism $f \colon X \to Y$ is called an equivalence if there is a morphism $g \colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* (or *contravariant functor*) $T$ from $\mathcal{C}$ to $\mathcal{D}$ consists of an object function which assigns to every object $X$ of $\mathcal{C}$ an object $T(X)$ of $\mathcal{D}$ and a morphism function which assigns to every morphism $f\colon X \to Y$ of $\mathcal{C}$ a morphism $T(f)\colon T(X) \to T(Y)$ [or $T(f)\colon T(Y) \to T(X)$] of $\mathcal{D}$ such that

a) $T(\mathrm{id}_X) = \mathrm{id}_{T(X)}$ and

b) $T(g \circ f) = T(g) \circ T(f)$ [or $T(g \circ f) = T(f) \circ T(g)$].

**Theorem 1.4.1** *Let $T$ be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. Then $T$ maps equivalences in $\mathcal{C}$ to equivalences in $\mathcal{D}$.*

*Proof.* Assume that $T$ is covariant (the argument is similar if $T$ is contravariant). Let $f\colon X \to Y$ be an equivalence and let $f^{-1}\colon Y \to X$ be its inverse. Since $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$, $T(f^{-1}) \circ T(f) = \mathrm{id}_{T(X)}$ and $T(f) \circ T(f^{-1}) = \mathrm{id}_{T(Y)}$. Thus $T(f)$ is an equivalence.

A topological problem on spaces is to how to classify topological spaces. In other words, roughly speaking, how to know whether a space $X$ is homeomorphic to another space $Y$ or not. Basic ideas in algebraic topology is to introduce various functors from the category of topological spaces to "algebraic" categories such as the category of groups, the category of abelian groups, and the category of modules and etc. Homology, fundamental group and higher homotopy groups are most important functors from the category of spaces to the category of groups.

For example, we will know that the fundamental group of $\mathbb{R} \setminus \{0\}$ is $\mathbb{Z}$ but the fundamental group of $\mathbb{R}^2 \setminus \{0\}$ is $\{0\}$. By Theorem 1.4.1, we have that $\mathbb{R} \setminus \{0\}$ is not homeomorphic to $\mathbb{R}^2 \setminus \{0\}$ and so $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. This is a simple example. Actually we will be able to classify all of (2-dimensional) surfaces in this course using the fundamental group.