

Moreover, the map g sends $(X \vee Y) \vee Z$ to the base point, so that g induces a map

$$\tilde{g}: (X \vee Y) \wedge Z \rightarrow (X \wedge Z) \vee (Y \wedge Z),$$

where $\tilde{g}((x, y_0) \wedge z) = x \wedge z$ in $X \wedge Z$ and $\tilde{g}((x_0, y) \wedge z) = y \wedge z$ in $Y \wedge Z$.

Conversely, let $h: (X \wedge Z) \vee (Y \wedge Z) \rightarrow (X \vee Y) \wedge Z$ be the map such that $h|_{X \wedge Z}$ and $h|_{Y \wedge Z}$ are the inclusions $X \wedge Z \hookrightarrow (X \wedge Y) \wedge Z$ and $Y \wedge Z \hookrightarrow (X \wedge Y) \wedge Z$, respectively. Then $h(x \wedge z) = (x, y_0) \wedge z$ and $h(y \wedge z) = (x_0, y) \wedge z$ so that $\tilde{g} \circ h$ and $h \circ \tilde{g}$ are identities, and hence \tilde{g} is a homeomorphism.

Exercise 2.6.3 Show that $S^n \wedge S^m \cong S^{n+m}$ for any n, m .

2.7 Topological Groups and Orbit Spaces

A pointed topological space X is called an H -space if there is a continuous multiplication $\mu: X \times X \rightarrow X$, $(x, y) \rightarrow xy$, such that $x_0x = xx_0 = x$. The base point x_0 is often denoted as $*$ or 1 . Equivalently, a pointed space X is an H -space if and only if there is a map $\mu: X \times X \rightarrow X$ such that $\mu|_{X \vee X} = \nabla$, where $\nabla: X \vee X \rightarrow X$ is the fold map defined by $\nabla(x, x_0) = x$ and $\nabla(x_0, x) = x$.

An H -space is called *associative* if diagram

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mu \times \text{id}_X} & X \times X \\ \downarrow \text{id}_X \times \mu & & \downarrow \mu \\ X \times X & \xrightarrow{\mu} & X \end{array}$$

is commutative. An associative H -space is called a *topological monoid*. In other words, a topological monoid is monoid as a set such that the multiplication is continuous. A *topological group* G means a topological monoid such that there is a map $\nu: G \rightarrow G$, $x \rightarrow x^{-1}$, with $xx^{-1} = 1 = x^{-1}x$, that is the inverse is a continuous function.

Let X be a space and let G be a topological group. We say that G *acts* on X and that X is a G -space if there is map $\mu: G \times X \rightarrow X$, denoted by $(g, x) \rightarrow g \cdot x$, such that

- i) $1 \cdot x = x$ for all $x \in X$;

ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$, that is the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\text{id}_G \times \mu} & G \times X \\ \downarrow \mu_G \times \text{id}_X & & \downarrow \mu \\ G \times X & \xrightarrow{\mu} & X \end{array}$$

commutes.

Theorem 2.7.1 *Suppose that X is a G -space. Then the function $\theta_g: X \rightarrow X$ given by $x \rightarrow g \cdot x$ is a homeomorphism. It follows that there is a homomorphism from G to the group of homeomorphisms of X .*

Proof. The function θ_g is the composite

$$X \cong \{g\} \times X \subseteq G \times X \xrightarrow{\mu} X.$$

Thus θ_g is continuous. From the definition of G -space we see that $\theta_g \circ \theta_h = \theta_{gh}$ and $\theta_1 = \text{id}_X$. Thus $\theta_g \circ \theta_{g^{-1}} = \text{id}_X = \theta_{g^{-1}} \circ \theta_g$ and so θ_g is a homeomorphism. Now the function $g \rightarrow \theta_g$ is a homomorphism from G to the group of homeomorphisms of X .

Let X be a G -space. We can define an equivalence relation \sim on X by

$$x \sim y \Leftrightarrow g \cdot x = y \text{ for some } g \in G.$$

The quotient space X/\sim , denoted by X/G , with the quotient topology is called the *quotient space* of X by G .

Example 2.7.2 1) Let $G = \mathbb{Z}/2 = \{\pm 1\}$ with discrete topology and let $X = S^n$. The G -action on X is given by $\pm 1 \cdot x = \pm x$. Then $S^n/(\mathbb{Z}/2) \cong \mathbb{R}P^n$.

2) Let $G = \mathbb{Z}$ with the discrete topology and let $X = \mathbb{R}$. The action of G on \mathbb{R} is given by $n \cdot x = n + x$. Then $\mathbb{R}/\mathbb{Z} \cong S^1$.

3) Let $G = S^1 \subseteq \mathbb{C}$. Then G is a topological group under the multiplication. Let $S^{2n-1} \subseteq \mathbb{R}^{2n} = \mathbb{C}^n$ be the unit sphere. Let G act on S^{2n-1} by

$$\alpha \cdot (z_1, z_2, \dots, z_n) = (\alpha z_1, \alpha z_2, \dots, \alpha z_n).$$

Then $S^{2n-1}/S^1 \cong \mathbb{C}P^n$.

- 4) Let M_n be the set of $n \times n$ -matrices over \mathbb{R} . Then $M_n = \mathbb{R}^{n^2}$ is a topological space. Let

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M_n \mid \det(A) \neq 0\} \subseteq M_n$$

with the subspace topology. Then $\mathrm{GL}(n, \mathbb{R})$ is a topological group, which called the *general linear group*.

- 5) Let $O(n)$ be the group of (real) orthogonal $n \times n$ matrices. $O(n)$ is regarded as a subspace of \mathbb{R}^{n^2} with the subspace topology. For $k \leq n$ $O(k)$ is regarded as the set of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

with A an orthogonal $k \times k$ -matrix and I_{n-k} the $(n - k) \times (n - k)$ identity matrix. Then $O(k)$ is a topological subgroup of $O(n)$. In $O(n)$ we also have the subgroup $SO(n)$ of orthogonal matrices with determinant 1, that is $SO(n)$ is the kernel of $\det: O(n) \rightarrow \mathbb{Z}/2$.

- 6) Let $U(n)$ denote the group of $n \times n$ unitary matrices regarded as a subspace of \mathbb{C}^{n^2} . We have the inclusions

$$U(1) \subseteq U(2) \subseteq U(3) \subseteq \cdots \subseteq U(n) \subseteq \cdots$$

Thus $U(k)$ is a topological subgroup of $U(n)$ for $k \leq n$. We also have the subgroup $SU(n) \subseteq U(n)$ of $n \times n$ unitary matrices with determinant 1, that is $SU(n)$ is the kernel of $\det: U(n) \rightarrow S^1$.

Theorem 2.7.3 *Suppose that X is a G -space. Then the canonical projection $\pi: X \rightarrow X/G$ is an open mapping.*

Proof. Let U be an open set in X . Then

$$\begin{aligned} \pi^{-1}(\pi(U)) &= \{x \in X \mid \pi(x) \in \pi(U)\} \\ &= \{x \in X \mid x = g \cdot y \text{ for some } y \in U \text{ some } g \in G\} = \bigcup_{g \in G} g \cdot U. \end{aligned}$$

Since $\theta_g: X \rightarrow X$ is a homeomorphism for each $g \in G$, $g \cdot U$ is open for each g then so $\pi^{-1}(\pi(U))$ is open and hence $\pi(U)$ is open in X/G .

Exercise 2.7.1 1) Let X be a G -space and define the *stabilizer* of $x \in X$ to be the subspace

$$G_x = \{g \in G \mid g \cdot x = x\}$$

of G . Show that G_x is a topological subgroup of G .

2) Let X be a G -space and define the *orbit* of $x \in X$ to be the subspace

$$G \cdot x = \{g \cdot x \mid g \in G\}$$

of X . Prove that $G \cdot x$ and $G \cdot y$ are either disjoint or equal for any $x, y \in X$.

2.8 Compact Spaces, Hausdorff Spaces and Locally Compact Spaces

Let X be a space. A *cover* of a subset S is a collection of subsets $\{U_j \mid j \in J\}$ of X such that

$$S \subseteq \bigcup_{j \in J} U_j.$$

A cover is called *finite* if the indexing set J is finite. Let $\{U_j \mid j \in J\}$ and $\{V_k \mid k \in K\}$ be covers of the subset S of X . $\{U_j \mid j \in J\}$ is called a *subcover* of $\{V_k \mid k \in K\}$ if

$$\{U_j \mid j \in J\} \subseteq \{V_k \mid k \in K\}.$$

Definition 2.8.1 Let X be a space. A subset S is called to be *compact* if every open cover of S has a finite subcover. In particular, a space X is compact if every open cover of X has a finite subcover.

Exercise 2.8.1 Show that a subset S of a space X is compact if and only if it is compact as a space given the induced topology.

Exercise 2.8.2 Show that $[0, 1] \subseteq \mathbb{R}$ is compact.

The following theorem is useful.

Theorem 2.8.2 Let $f: X \rightarrow Y$ be a map. If $S \subseteq X$ is a compact subspace, then $f(S)$ is compact.

Proof. Suppose that $\{U_j|j \in J\}$ be an open cover of $f(S)$. Then $\{f^{-1}(U_j)|j \in J\}$ is an open cover of S . Since S is compact, there exists a finite subset K of J such that

$$S \subseteq \bigcup\{f^{-1}(U_k)|k \in K\}.$$

But $f(f^{-1}(U_k)) \subseteq U_k$ and so

$$f(S) \subseteq \bigcup\{f(f^{-1}(U_k))|k \in K\} \subseteq \{U_k|k \in K\}$$

which is a finite subcover of $\{U_j|j \in J\}$.

Theorem 2.8.3 *A closed subset of a compact space is compact.*

Proof. Let X be a compact space and let S be a closed subset of X . Let $\{U_j\}$ be an open cover of S . Since $S \subseteq \bigcup\{U_j|j \in J\}$ we see that

$$X \subseteq \bigcup\{U_j|j \in J\} \cup (X \setminus S)$$

and so there is a finite subcover

$$X \subseteq \bigcup\{U_k|k \in K\} \cup (X \setminus S).$$

Thus

$$S \subseteq \bigcup\{U_k|k \in K\}$$

which is a finite subcover of $\{U_j|j \in J\}$.

Theorem 2.8.4 *Let X and Y be spaces. Then X and Y are compact if and only if $X \times Y$ is compact.*

Proof. Suppose that $X \times Y$ is compact. Since $\pi_X: X \times Y \rightarrow X$ and $\pi_Y: X \times Y \rightarrow Y$ are continuous, X and Y are compact. Conversely assume that X and Y are compact. Let $\{W_j|j \in J\}$ be an open cover of $X \times Y$. By definition

$$W_j = \bigcup_{k \in K(j)} (U_{j,k} \times V_{j,k})$$

where $U_{j,k}$ and $V_{j,k}$ are open in X and Y , respectively. Thus

$$X \times Y \subseteq \bigcup_{j \in J, k \in K(j)} U_{j,k} \times V_{j,k}.$$

For each $x \in X$ the subspace $\{x\} \times Y$ is compact and so there is a finite subcover

$$\{x\} \times Y \subseteq \bigcup_{i=1}^{n(x)} U_i(x) \times V_i(x).$$

Let $U'(x) = \bigcap_{i=1}^{n(x)} U_i(x)$. Then $U'(x)$ is an open neighborhood of x and

$$X \subseteq \bigcup_{x \in X} U'(x)$$

Since X is compact, there are finite points x_1, \dots, x_m such that

$$X \subseteq \bigcup_{j=1}^m U'(x_j).$$

It follows that

$$X \times Y \subseteq \bigcup_{1 \leq j \leq m, 1 \leq i \leq n(x_j)} U'(x_j) \times V_i(x_j) \subseteq \bigcup_{1 \leq j \leq m, 1 \leq i \leq n(x_j)} U_i(x_j) \times V_i(x_j).$$

Since for each $U_i(x_j) \times V_i(x_j)$ there is an index k such that

$$U_i(x_j) \times V_i(x_j) \subseteq W_k,$$

there is a finite subcover of $\{W_j | j \in J\}$ covering $X \times Y$.

A space X is called *Hausdorff* if for every pair of distinct points x and y there are open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. In other words, X is Hausdorff if for any $x \neq y$ in X there are neighborhood $N(x)$ and $N(y)$ of x and y , respectively such that $N(x) \cap N(y) = \emptyset$. Hausdorff space is also called T_2 -space. In a Hausdorff space X , any point x is a closed subset. (This is not true for general topological space. For example, the indiscrete topology.)

Theorem 2.8.5 *A compact subset A of a Hausdorff space X is closed.*

Proof. We may assume that $A \neq \emptyset$ and $A \neq X$. Given $x \in X \setminus A$. For each $a \in A$, there are disjoint open sets $U_a(x)$ and $V_a(x)$ such that $a \in U_a(x)$ and $x \in V_a(x)$. Since

$$A \subseteq \bigcup_{a \in A} U_a(x)$$

and A is compact, there are finite points a_1, \dots, a_m in A such that

$$A \subseteq \bigcup_{i=1}^m U_{a_i}(x).$$

Now the set $V(x) = \bigcap_{i=1}^m V_{a_i}(x)$ is an open neighborhood of x with

$$A \cap V(x) \subseteq \left(\bigcup_{i=1}^m U_{a_i}(x) \right) \cap V(x) = \emptyset$$

and so $V(x) \subseteq X \setminus A$, which means that $X \setminus A$ is open or A is closed.

In particular, if X can be embedded into \mathbb{R}^n then X must be Hausdorff.

Theorem 2.8.6 (Heine-Borel) *A subset S of \mathbb{R}^n is compact if and only if it is closed and bounded.*

Proof. Suppose that S is compact. By Theorem 2.8.5, S is closed. Now

$$S \subseteq \bigcup_{x \in S} B_1(x)$$

and so there exist finite points x_1, \dots, x_m in S such that

$$S \subseteq \bigcup_{i=1}^m B_1(x_i).$$

Thus S is bounded. Conversely suppose that S is closed and bounded. There exists positive number $r \gg 0$ such that

$$S \subseteq [-r, r]^n.$$

Since $[-r, r]$ is compact, $[-r, r]^n$ is compact. By Theorem 2.8.3, the closed subspace S is compact.

Exercise 2.8.3 Let X and Y be spaces. Then X and Y are Hausdorff if and only if $X \times Y$ is Hausdorff.

Thus the spaces like n -Torus $T^n = S^1 \times S^1 \times \dots \times S^1$ are (compact) Hausdorff.

Exercise 2.8.4 Let X and Y be topological spaces. Show that

- 1) If X is Hausdorff, then any subspace of X is Hausdorff;
- 2) X and Y are Hausdorff if and only if $X \times Y$ is Hausdorff;
- 3) X is Hausdorff if and only if the diagonal $\Delta(X) = \{(x, x) \in X^2 | x \in X\}$ is a closed subset of X^2 ;
- 4) $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$ are compact Hausdorff spaces;
- 5) Let $f: X \rightarrow Y$ be a map. Suppose that X is compact Hausdorff and Y is Hausdorff. Then f is a closed map. Reduce that a bijective map from a compact Hausdorff space to a Hausdorff space is a homeomorphism.

A quotient map $f: X \rightarrow Y$ is also called *identification* map. A quotient space may not be Hausdorff.

Theorem 2.8.7 *Let $f: X \rightarrow Y$ be an identification map. Suppose that X is Hausdorff. If f is closed and $f^{-1}(y)$ is compact for any $y \in Y$, then Y is Hausdorff.*

Proof. Let y_1 and y_2 be distinct points in Y . For each $x \in f^{-1}(y_1)$ and $a \in f^{-1}(y_2)$, there exist a pair of disjoint open sets $U_{x,a}$ and $V_{x,a}$ with $x \in U_{x,a}$ and $a \in V_{x,a}$. Fixed $x \in f^{-1}(y_1)$ $\{V_{x,a} | a \in f^{-1}(y_2)\}$ is an open cover of $f^{-1}(y_2)$. By the assumption, $f^{-1}(y_2)$ is compact and so there are finite points $a_1(x), \dots, a_{m(x)}(x)$ such that

$$f^{-1}(y_2) \subseteq \bigcup_{i=1}^{m(x)} V_{x,a_i(x)}.$$

Let $V(x) = \bigcup_{i=1}^{m(x)} V_{x,a_i(x)}$ and let $U(x) = \bigcap_{i=1}^{m(x)} U_{x,a_i(x)}$. Then $U(x)$ is an open neighborhood of x and $V(x)$ is an open neighborhood of $f^{-1}(y_2)$ with $U(x) \cap V(x) = \emptyset$. Since $f^{-1}(y_1) \subseteq \bigcup_{x \in f^{-1}(y_1)} U(x)$ and $f^{-1}(y_1)$ is compact, there are finite points $x_1, \dots, x_s \in f^{-1}(y_1)$ such that

$$f^{-1}(y_1) \subseteq \bigcup_{j=1}^s U(x_j).$$

Let $U = \bigcup_{j=1}^s U(x_j)$ and $V = \bigcap_{j=1}^s V(x_j)$. Then U and V are disjoint open sets with $f^{-1}(y_1) \subseteq U$ and $f^{-1}(y_2) \subseteq V$. Since f is closed, $f(X \setminus U)$ and $f(X \setminus V)$ are closed subsets in Y and so $W_1 = Y \setminus f(X \setminus U)$ and $W_2 = Y \setminus f(X \setminus V)$ are open subsets in Y with $y_1 \in W_1$ and $y_2 \in W_2$. We show that $W_1 \cap W_2 = \emptyset$. Suppose that $y \in W_1 \cap W_2$. Then $y \notin f(X \setminus U)$ and $y \notin f(X \setminus V)$. Therefore $f^{-1}(y) \cap (X \setminus U) = \emptyset$ and $f^{-1}(y) \cap (X \setminus V) = \emptyset$. It follows that $f^{-1}(y) \subseteq U \cap V = \emptyset$ and hence $W_1 \cap W_2 = \emptyset$.

Corollary 2.8.8 *Let X be a compact Hausdorff space. Then*

- i) If G is a finite group and X is a G -space, then X/G is a compact Hausdorff space;*
- ii) If A is closed subspace of X , then X/A is compact Hausdorff.*

Exercise 2.8.5 A space is called *normal* (T_4 -space) if every point in X is closed and every pair of disjoint closed sets has disjoint open neighborhood. Let G be a compact topological space and let X be a normal G -space. Show that X/G is Hausdorff.

(Hint: Let $\pi: X \rightarrow X/G$ be the quotient map. For each $y \in X/G$, $\pi^{-1}(y) = G \cdot x$ for some $x \in X$ with $\pi(x) = y$. Show that the orbit $G \cdot x$ is a quotient of G . Since G is compact, the orbit $G \cdot x$ is compact and so it is closed because T_4 -space is Hausdorff. Let $y_1 \neq y_2$ be distinct points in X/G . Then $\pi^{-1}(y_1)$ and $\pi^{-1}(y_2)$ are disjoint closed set and so they have disjoint open neighborhood, say U and V . By Theorem 2.7.3, $\pi(U)$ and $\pi(V)$ are disjoint open neighborhoods of y_1 and y_2 , respectively.)

For example, $\mathbb{R}P^n$ is compact Hausdorff space because $\mathbb{R}P^n$ is the quotient of S^n by the action of $\mathbb{Z}/2$. $\mathbb{C}P^n$ is a compact Hausdorff space because it is the quotient of S^{2n+1} by S^1 .

A space X is called *locally compact* if every point x in X has a compact neighborhood.

Exercise 2.8.6 Let X be a locally compact Hausdorff space. Given a point $x \in X$ and a neighborhood U of x . Show that there is an open set V such that $x \in V \subseteq \bar{V} \subseteq U$ and \bar{V} is compact. (Hint: Let W be a compact neighborhood of x , that is there is an open set U_1 such that $x \in U_1 \subseteq W$ and W is compact. Let $V_1 = U_1 \cap U$. Then V_1 is an open neighborhood of x and $\bar{V}_1 \setminus V_1$ is compact because it is a closed subset of the compact space W . Let $A = \bar{V}_1 \setminus V_1$. For each $y \in A$, there exist disjoint open sets $U(y)$ and $V(y)$ such that $y \in U(y)$ and $x \in V(y)$ because X is Hausdorff. Since A is compact and $A \subseteq \bigcup_{y \in A} U(y)$, there are finite points y_1, \dots, y_n such that $A \subseteq \bigcup_{i=1}^n U(y_i)$. Let

$$V = V_1 \cap \bigcap_{i=1}^n V(y_i).$$

Then V is an open neighborhood of x with

- 1) $\bar{V} \cap A = \emptyset$ (because V is disjoint with an open neighborhood, $\bigcup_{i=1}^n U(y_i)$, of A ;

2) $\bar{V} \subseteq \bar{V}_1$ because $V \subseteq V_1$ and

3) \bar{V} is compact because it is a closed subset of \bar{V}_1 .

By 1) and 2) above, we have that $\bar{V} \subseteq V_1 \subseteq U$.

Theorem 2.8.9 (a) *If $p: X \rightarrow Y$ is a quotient map and Z is a locally compact Hausdorff space, then $p \times \text{id}_Z: X \times Z \rightarrow Y \times Z$ is a quotient map.*

(b) *If A is a compact subspace of a space X and $p: X \rightarrow X/A$ is the quotient map, then for any space Z , $p \times \text{id}_Z: X \times Z \rightarrow (X/A) \times Z$ is a quotient map.*

Proof. Let $\pi = p \times \text{id}_Z$.

(a) Let A be a subset of $Y \times Z$ such that $\pi^{-1}(A)$ is open in $X \times Z$. We show that A is open. Let $(y_0, z_0) \in Y \times Z$. Choose $x_0 \in X$ such that $p(x_0) = y_0$.

Since $\pi^{-1}(A)$ is open and Z is locally compact, there are open sets U_1 in X and V in Z such that \bar{V} is compact, $U_1 \times V$ is an open neighborhood of (x_0, z_0) and $U_1 \times \bar{V} \subseteq \pi^{-1}(A)$. The point here is that $p^{-1}(p(U_1))$ is not necessarily open in X but it contains U_1 . We do the following construction.

Suppose that U_i is an open neighborhood of x_0 such that $U_i \times \bar{V} \subseteq (p \times \text{id}_Z)^{-1}(A)$. We construct an open set U_{i+1} of X such that

$$p^{-1}(p(U)) \times \bar{V} \subseteq U_{i+1} \times \bar{V} \subseteq \pi^{-1}(A),$$

as follows: For each point $x \in p^{-1}(p(U_i))$ the space $\{x\} \times \bar{V}$ lies in $\pi^{-1}(A)$. Using compactness of \bar{V} , we choose a neighborhood W_x of x such that $W_x \times \bar{V} \subseteq \pi^{-1}(A)$. Let U_{i+1} be the union of the open sets W_x ; then U_{i+1} is the desired open set of X .

Finally, let U be the union of the open sets $U_1 \subseteq U_2 \subseteq \dots$. Then $U \times V$ is a neighborhood of (x_0, z_0) and $U \times \bar{V} \subseteq \pi^{-1}(A)$. Since

$$U \subseteq p^{-1}(p(U)) = p^{-1}\left(\bigcup_{i=1}^{\infty} p(U_i)\right) = \bigcup_{i=1}^{\text{infy}} p^{-1}(p(U_i)) \subseteq \bigcup_{i=1}^{\infty} U_{i+1} = U,$$

we have $p^{-1}(p(U)) = U$ and so $p(U)$ is open in Y . Thus

$$p(U) \times V = \pi(U \times V) \subseteq A$$

is a neighborhood of (x_0, z_0) lying in A , as desired.

(b) Again it suffices to show that a subset U in $X/A \times Z$ is open if $\pi^{-1}(U)$ is open in $X \times Z$. As in case (a), let $(y_0, z_0) \in U$ and let $x_0 \in X$ such that $p(x_0) = y_0$.

If $x_0 \in A$, then $A \times \{z_0\} \subseteq \pi^{-1}(U)$. Since A is compact, a similar argument to that used in case (a) shows that there exist open sets $V \subseteq X$ and $W \subseteq Z$ such that

$$A \times \{z_0\} \subseteq V \times W \subseteq \pi^{-1}(U).$$

But then $(y_0, z_0) \in p(V) \times W \subseteq U$; $p(V)$ is open since $p^{-1}(p(V)) = V$ (because $A \subseteq V$), and so $p(V) \times W$ is open.

If on the other hand $x \notin A$, there certainly exist open sets $V \subseteq X$ and $W \subseteq Z$ such that $(x_0, z_0) \in V \times W \subseteq \pi^{-1}(U)$ and if $V \cap A = \emptyset$, then $p(V) \times W$ is open. However, if $V \cap A \neq \emptyset$, then $(p(A), z_0) \in U$, and we have already seen that we can then write

$$(p(A), z_0) \in p(\tilde{V}) \times \tilde{W} \subseteq U.$$

But then $(y_0, z_0) \in p(V \cup \tilde{V}) \times (W \cap \tilde{W}) \subseteq U$; $p(V \cup \tilde{V})$ is open since $A \subseteq \tilde{V}$, and so once again (x_0, z_0) is contained in an open subset of U . It follows that U is open. ♠

Corollary 2.8.10 *If $p: A \rightarrow B$ and $q: C \rightarrow D$ are quotient maps and if the domain of p and the range of q are locally compact Hausdorff spaces, then*

$$p \times q: A \times B \rightarrow C \times D$$

is a quotient map.

Proof. We can write $p \times q$ as the composite

$$A \times B \xrightarrow{\text{id}_A \times q} A \times D \xrightarrow{p \times \text{id}_D} C \times D.$$

Since each of these maps is a quotient map, so is the composite $p \times q$. ♠

Theorem 2.8.11 *If X and Y are compact and X is Hausdorff, then $(X \wedge Y) \wedge Z$ is homeomorphic to $X \wedge (Y \wedge Z)$.*

Proof. Write p for the various quotient maps of the form $X \times Y \rightarrow X \wedge Y$, and consider the diagram

$$\begin{array}{ccc} X \times Y \times Z & \xlongequal{\quad} & X \times Y \times Z \\ \downarrow p \times \text{id}_Z & & \downarrow \text{id}_X \times p \\ (X \wedge Y) \times Z & & X \times (Y \wedge Z) \\ \downarrow p & & \downarrow p \\ (X \wedge Y) \wedge Z & & X \wedge (Y \wedge Z). \end{array}$$

Since X and Y are compact, $X \vee Y$ is compact. By Theorem 2.8.9, the map

$$p \times \text{id}_Z: X \times Y \times Z \rightarrow (X \wedge Y) \times Z$$

is a quotient map. Since X is locally compact and Hausdorff, again by Theorem 2.8.9, the map $\text{id}_X \times p$ is a quotient map. It follows that both $p \circ (p \times \text{id}_Z)$ and $p \circ (\text{id}_X \times p)$ are quotient maps. The identity map $\text{id}: X \times Y \times Z \rightarrow X \times Y \times Z$ induces maps

$$f: (X \wedge Y) \wedge Z \rightarrow X \wedge (Y \wedge Z) \text{ and}$$

$$g: X \wedge (Y \wedge Z) \rightarrow (X \wedge Y) \wedge Z$$

that are clearly homeomorphisms. ♠

Exercise 2.8.7 Show that $X \wedge (Y \wedge Z)$ is homeomorphic to $X \wedge (Y \wedge Z)$ if X and Z are locally compact and Hausdorff.

2.9 Mapping Spaces and Compact-open Topology

Given spaces X and Y , the *mapping space* $\text{Map}(X, Y)$ consists of all (continuous) maps from X to Y . The topology in $\text{Map}(X, Y)$ is given by so-called *compact-open* topology that is defined as follows.

Let K be a compact set in X and let U be an open set in Y . Let

$$W_{K,U} = \{f \in \text{Map}(X, Y) \mid f(K) \subseteq U\}.$$

The compact-open topology in $\text{Map}(X, Y)$ is generated by $W_{K,U}$ where K runs over all compact subsets in X and U runs over all open sets in Y . In other words, an open set in $\text{Map}(X, Y)$ is a union of a finite intersection of the subsets with the form $W_{K,U}$.

If X and Y are pointed spaces. Then pointed mapping space, denoted by Y^X or $\text{Map}_*(X, Y)$, is the subspace of $\text{Map}(X, Y)$ consisting of all pointed (continuous) maps, that all of maps $f: X \rightarrow Y$ with $f(x_0) = y_0$.

Exercise 2.9.1 Let Y be a space and let X be a space with discrete topology. Show that the compact-open topology on

$$\text{Map}(X, Y) = \prod_{x \in X} Y_x,$$

where Y_x is a copy of Y , is the same as the product topology.

Let $f: A \rightarrow X$ and $g: Y \rightarrow B$ be maps. Then the function $g^f: \text{Map}(X, Y) \rightarrow \text{Map}(A, B)$ is defined by

$$g^f(\lambda) = g \circ \lambda \circ f$$

for $\lambda: X \rightarrow Y$. If f and g are pointed maps, then g^f induces the map $g^f: \text{Map}_*(X, Y) \rightarrow \text{Map}_*(A, B)$ because if $\lambda \in \text{Map}_*(X, Y)$, that is λ is a pointed map, then $g^f(\lambda)$ is a pointed map.

Proposition 2.9.1 *Let $f: A \rightarrow X$ and $g: Y \rightarrow B$ be [pointed] maps. Then $g^f: \text{Map}(X, Y) \rightarrow \text{Map}(A, B)$ [$g^f: \text{Map}_*(X, Y) \rightarrow \text{Map}_*(A, B)$] is continuous.*

Proof. Take a sub-basic open set $W_{K,U}$ in $\text{Map}(A, B)$, where K is compact in A and U is open in B . Then

$$\begin{aligned} (g^f)^{-1}(W_{K,U}) &= \{\lambda: X \rightarrow Y \mid g \circ \lambda f(K) \subseteq U\} \\ &= \{\lambda: X \rightarrow Y \mid \lambda(f(K)) \subseteq g^{-1}(U)\} = W_{f(K), g^{-1}(U)} \end{aligned}$$

because $f(K)$ is compact in X and $g^{-1}(U)$ is open in Y . Thus g^f is continuous. ♠

Let X and Y be pointed spaces. Then both $\text{Map}_*(X, Y)$ and $\text{Map}(X, Y)$ are pointed spaces, where the base-point is the constant map $c: X \rightarrow Y$, $c(x) = y_0$. Let $i: \{x_0\} \rightarrow X$ be the inclusion, then we have the sequence

$$\text{Map}_*(X, Y) \hookrightarrow \text{Map}(X, Y) \xrightarrow{\text{id}_Y^i} \text{Map}(\{x_0\}, Y) \cong Y.$$

This sequence is called the canonical fibration for mapping spaces. Observe that $\lambda \in \text{Map}_*(X, Y)$ is and only if $\text{id}_Y^i(\lambda)$ is the base-point. As sets, one can see that $\text{Map}(X, Y)$ is isomorphic to $\text{Map}_*(X, Y) \times Y$. But as spaces $\text{Map}(X, Y)$ is quite different from $\text{Map}_*(X, Y) \times Y$ in general. The pointed mapping space $\text{Map}_*(S^1, Y)$ is denoted by ΩY , which is called the *loop space* of Y . The mapping space $\text{Map}(S^1, Y)$ is often denoted by ΛY and is call the *free loop space* of Y in many references. We will see that ΩY is actually an H -space, while ΛY is not in general. It was found in physics that some problems related to so-called *n-body* problem in physics are related to the homology of ΛY for certain spaces Y . There are many machines in algebraic topology for computing the homology of ΩY , but the determination of the homology of ΛY for many interesting spaces Y remains as interesting problems and have been studied by people.

Proposition 2.9.2 (a) *If Z is a subspace of Y , then $\text{Map}(X, Z)$ is a subspace of $\text{Map}(X, Y)$;*

(b) If Z is a pointed subspace of Y , then $\text{Map}_*(X, Z)$ is a subspace of $\text{Map}_*(X, Y)$.

Proof. The proofs of assertions (a) and (b) are similar. So we only prove assertion (a). We have to show that a set is open in $\text{Map}(X, Z)$ if and only if it is the intersection with $\text{Map}(X, Z)$ of a set that is open in $\text{Map}(X, Y)$. Let $j: Z \rightarrow Y$ be the inclusion. Then $j^{\text{id}_X}: \text{Map}(X, Z) \rightarrow \text{Map}(X, Y)$ is continuous, so that if $U \subseteq \text{Map}(X, Y)$ is open, $U \cap \text{Map}(X, Z) = (j^{\text{id}_X})^{-1}(U)$ is open in $\text{Map}(X, Z)$. To prove the converse, it is sufficient to consider an open set in $\text{Map}(X, Z)$ of the form $W_{K,U}$, where $K \subseteq X$ is compact and $U \subseteq Z$ is open. But $U = V \cap Z$ for some open set V in Y and

$$\begin{aligned} W_{K,V} \cap \text{Map}(X, Z) &= \{f: X \rightarrow Y \mid f(K) \subseteq V \text{ and } f(X) \subseteq Z\} \\ &= \{f: X \rightarrow Z \mid f(K) \subseteq V \cap Z = U\} = W_{K,U}. \end{aligned}$$

That is an open set in $\text{Map}(X, Z)$ is the intersection with $\text{Map}(X, Z)$ of an open set in $\text{Map}(X, Y)$. ♠

Given spaces X and Y , the *evaluation map*

$$e: \text{Map}(X, Y) \times X \rightarrow Y$$

is defined by

$$e(\lambda, x) = \lambda(x)$$

for $x \in X$ and $\lambda: X \rightarrow Y$. If X and Y are pointed spaces, the restriction of e gives the evaluation map $e: \text{Map}_*(X, Y) \times X \rightarrow Y$. If λ is the constant map or x is the base point x_0 , then $e(\lambda, x) = y_0$. That is $e(\text{Map}_*(X, Y) \vee X) = y_0$ and so e induces the *evaluation map*

$$e: \text{Map}_*(X, Y) \wedge X \rightarrow Y.$$

Theorem 2.9.3 *Let X and Y be pointed spaces. If X is locally compact Hausdorff, then the evaluation maps*

$$e: \text{Map}(X, Y) \times X \rightarrow Y \text{ and}$$

$$e: \text{Map}_*(X, Y) \wedge X \rightarrow Y$$

are continuous.

Proof. Let U be an open set in Y and that $e(\lambda, x) = \lambda(x) \in U$. Then $x \in \lambda^{-1}(U)$ which is open in X . Since X is locally compact and Hausdorff, there exists an open set V in X such that $x \in V \subseteq \bar{V} \subseteq \lambda^{-1}(U)$, and \bar{V} is compact. Consider

$W_{\bar{V},U} \times V \subseteq \text{Map}(X, Y) \times X$; this contains (λ, x) and if (λ', x') is another point in it, then

$$e(\lambda', x') = \lambda'(x') \in \lambda'(\bar{V}) \subseteq U.$$

Thus $W_{\bar{V},U} \times V \subseteq e^{-1}(U)$ and so $e^{-1}(U)$ is open or $e: \text{Map}(X, Y) \times X \rightarrow Y$ is continuous. It follows that the restriction

$$e: \text{Map}_*(X, Y) \times X \rightarrow Y$$

is continuous and so $e: \text{Map}_*(X, Y) \wedge X \rightarrow Y$ is continuous. ♠

Note: The evaluation $e: \text{Map}(X, Y) \times X \rightarrow Y$ may NOT be continuous in general. This is somewhat “not-so-good” in the category of topological spaces. Norman Steenrod then introduced “compact generated topological spaces” as a convenient category of topological spaces [4]. We just give the definition of compactly generated space. A space X is called *compactly generated* if X is Hausdorff and each subset A of X with the property that $A \cap C$ is closed for every compact subset C of X is itself closed. A locally compact Hausdorff space is compactly generated.

Theorem 2.9.4 *Let X, Y and Z be pointed spaces. Suppose that X and Y are Hausdorff. Then*

- (a) $\text{Map}(X \amalg Y, Z) \cong \text{Map}(X, Z) \times \text{Map}(Y, Z)$;
- (b) $\text{Map}_*(X \vee Y, Z) \cong \text{Map}_*(X, Z) \times \text{Map}_*(Y, Z)$.

Proof. We only prove assertion (b). Let x_0 and y_0 are base points of X and Y respectively, and define

$$i_X: X \rightarrow X \vee Y, \quad i_Y: Y \rightarrow X \vee Y$$

by $i_X(x) = (x, y_0)$ and $i_Y(y) = (x_0, y)$. Then i_X and i_Y are continuous. Define a function

$$\theta: \text{Map}_*(X, Z) \times \text{Map}_*(Y, Z) \rightarrow \text{Map}_*(X \vee Y, Z \vee Z)$$

by $\theta(\lambda, \mu) = \lambda \vee \mu$ for $\lambda: X \rightarrow Z$ and $\mu: Y \rightarrow Z$. Consider the composites

$$\begin{aligned} \phi: Z^{X \vee Y} &\xrightarrow{\Delta} Z^{X \vee Y} \times Z^{X \vee Y} \xrightarrow{\text{id}_Z^{i_X} \times \text{id}_Z^{i_Y}} Z^X \times Z^Y \quad \text{and} \\ \psi: Z^X \times Z^Y &\xrightarrow{\theta} (Z \vee Z)^{X \vee Y} \xrightarrow{\nabla} Z^{X \vee Y}, \end{aligned}$$

where Δ is the diagonal map and $\nabla: Z \vee Z \rightarrow Z$ is the fold map, that is $\nabla(z, z_0) = \nabla(z_0, z) = z$ for $z \in Z$. Given $\nu: X \vee Y \rightarrow Z$, $\phi(\nu) = (\nu \circ i_X, \nu \circ i_Y)$ and given

$\lambda: X \rightarrow Z$ and $\mu: Y \rightarrow Z$, $\psi(\lambda, \mu) = \nabla(\lambda \vee \mu)$. Thus $\phi \circ \psi$ and $\psi \circ \phi$ are identity functions, and the only point that remains in showing ϕ is a homeomorphism is to show that θ is continuous.

To do so, consider the set $W_{K,U}$, where $K \subseteq X \vee Y$ is compact and $U \subseteq Z \vee Z$ is open. Now

$$\begin{aligned} \theta^{-1}(W_{K,U}) &= \{(\lambda, \mu) \mid (\lambda \vee \mu)(K) \subseteq U\} \\ &= \{(\lambda, \mu) \mid \lambda(K \cap X) \subseteq U \cap (Z \times \{z_0\}) \text{ and } \mu(K \cap Y) \subseteq U \cap (\{z_0\} \times Z)\}. \end{aligned}$$

Clearly $U_1 = U \cap (Z \times \{z_0\})$ and $U_2 = U \cap (\{z_0\} \times Z)$ are open. But since X and Y are Hausdorff, so is $X \times Y$ and hence is $X \vee Y$; thus K, X, Y are closed in $X \vee Y$, so that $K \cap X$ and $K \cap Y$ are closed and hence compact. That is,

$$\theta^{-1}(W_{K,U}) = W_{K \cap X, U_1} \times W_{K \cap Y, U_2}$$

so that θ is continuous and hence ϕ is a homeomorphism. ♠

Let X be a topological space and let \mathcal{S} be a family of subsets of X . \mathcal{S} is called a *sub-base* of open sets if any member in \mathcal{S} is open and any open set in X is a union of finite intersections of members in \mathcal{S} . In other words if \mathcal{S} is a sub-base of open sets then the topology on X is generated by \mathcal{S} . We are going to give a result involving $(Y \times Z)^X$ and $Y^X \times Z^X$. We need the following lemma.

Lemma 2.9.5 *Let X be a Hausdorff space and let \mathcal{S} be a sub-base of open sets for a space Y . Then the sets of the form $W_{K,U}$ for $K \subseteq X$ compact and $U \in \mathcal{S}$, form a sub-base of open sets for $\text{Map}(X, Y)$.*

Proof. Let $K \subseteq X$ be compact, $V \subseteq Y$ be open and let $\lambda \in W_{K,V}$. Then $V = \bigcup_{\alpha} V_{\alpha}$, where V_{α} is a finite intersection of members in \mathcal{S} , and so

$$K \subseteq \bigcup_{\alpha} \lambda^{-1}(V_{\alpha});$$

hence, since K is compact, a finite collection of the sets $\lambda^{-1}(V_{\alpha})$, say $\lambda^{-1}(V_1), \dots, \lambda^{-1}(V_n)$, suffice to cover K . Given $x \in K$, there exists r such that $x \in \lambda^{-1}(V_r)$. Since K is a compact Hausdorff space and $K \cap \lambda^{-1}(V_r)$ is an open neighborhood of x , there exists an open set A_x in K such that

$$x \in A_x \subseteq \bar{A}_x \subseteq K \cap \lambda^{-1}(V_r).$$

Again, a finite collection of the open sets A_x will cover K , and their closures are each contained in just one set of the form $\lambda^{-1}(V_r)$. Thus by taking suitable unions of \bar{A}_x 's,

we can write $K = \bigcup_{r=1}^n K_r$, where $K_r \subseteq \lambda^{-1}(V_r)$ and K_r is closed and so compact. It follows that

$$\lambda \in \bigcap_{r=1}^n W_{K_r, V} \subseteq W_{K, V},$$

since if $\mu(K_r) \subseteq V_r$, for each r , then $\mu(K) \subseteq \bigcup_{r=1}^n V_r \subseteq V$. But if, say, $V_r = \bigcup_{s=1}^m U_s$ for $U_s \in \mathcal{S}$, then $W_{K_r, V_r} = \bigcap_{s=1}^m W_{K_r, U_s}$. Hence λ is contained in a finite intersection of sets of the form W_{K_r, U_s} for $U_s \in \mathcal{S}$ and this intersection is contained in $W_{K, V}$. ♠

Theorem 2.9.6 *Let X, Y and Z be pointed spaces. Suppose that X is Hausdorff. Then*

$$\begin{aligned} \text{Map}(X, Y \times Z) &\cong \text{Map}(X, Y) \times \text{Map}(X, Z) \quad \text{and} \\ \text{Map}_*(X, Y \times Z) &\cong \text{Map}_*(X, Y) \times \text{Map}_*(X, Z). \end{aligned}$$

Proof. We only prove that

$$\text{Map}(X, Y \times Z) \cong \text{Map}(X, Y) \times \text{Map}(X, Z).$$

Let $p_Y: Y \times Z \rightarrow Y$ and $p_Z: Y \times Z \rightarrow Z$ be coordinate projections. Define a function

$$\theta: \text{Map}(X, Y) \times \text{Map}(X, Z) \rightarrow \text{Map}(X \times X, Y \times Z)$$

by $\theta(\lambda, \mu) = \lambda \times \mu$ for $\lambda: X \rightarrow Y$ and $\mu: X \rightarrow Z$. Consider the composites

$$\phi: \text{Map}(X, Y \times Z) \xrightarrow{\Delta} \text{Map}(X, Y \times Z) \times \text{Map}(X, Y \times Z) \xrightarrow{p_Y^{\text{id}_X} \times p_Z^{\text{id}_X}} \text{Map}(X, Y) \times \text{Map}(X, Z)$$

$$\psi: \text{Map}(X, Y) \times \text{Map}(X, Z) \xrightarrow{\theta} \text{Map}(X \times X, Y \times Z) \xrightarrow{\text{id}^\Delta} \text{Map}(X, Y \times Z),$$

where Δ is a diagonal map. If $\nu: X \rightarrow Y \times Z$, then $\phi(\nu) = (p_Y \circ \nu, p_Z \circ \nu)$ and if $\lambda: X \rightarrow Y$ and $\mu: X \rightarrow Z$, then $\psi(\lambda, \mu) = (\lambda \times \mu) \circ \Delta$. Thus $\phi \circ \psi$ and $\psi \circ \phi$ are identity functions, and it remains only to prove that θ is continuous.

Since X is Hausdorff, by Lemma 2.9.5, it is sufficient to consider sets of the form $W_{K, U \times V}$, where $K \subseteq X \times X$ is compact and $U \subseteq Y, V \subseteq Z$ are open. Then

$$\theta^{-1}(W_{K, U \times V}) = \{(\lambda, \mu) \mid (\lambda \times \mu)(K) \subseteq U \times V\} = \{(\lambda, \mu) \mid K \subseteq \lambda^{-1}(U) \times \mu^{-1}(V)\}.$$

But if $p_1, p_2: X \times X \rightarrow X$ be the first and the second coordinate projections, then $p_1(K)$ and $p_2(K)$ are compact, and $K \subseteq \lambda^{-1}(U) \times \mu^{-1}(V)$ if and only if $p_1(K) \times p_2(K) \subseteq \lambda^{-1}(U) \times \mu^{-1}(V)$. Hence

$$\theta^{-1}(W_{K, U \times V}) = W_{p_1(K), U} \times W_{p_2(K), V}$$

and so θ is continuous. ♠

At this point we possess rules for manipulating mapping spaces analogous to the index laws $a^{b+c} = a^b \cdot a^c$ and $(a \cdot b)^c = a^c \cdot b^c$ for real numbers, and it remains to investigate what rule, if any, corresponds to the index law $a^{b \cdot c} = (a^b)^c$. Now we define the ‘association map’.

Given spaces X , Y and Z , the (*unreduced*) *association map* is the function $\alpha: \text{Map}(X \times Y, Z) \rightarrow \text{Map}(X, \text{Map}(Y, Z))$ defined by

$$[\alpha(\lambda)(x)](y) = \lambda(x, y)$$

for $x \in X$, $y \in Y$ and $\lambda: X \times Y \rightarrow Z$.

To justify this definition, we have to show that $\alpha(\lambda)$ is an element in $\text{Map}(X, \text{Map}(Y, Z))$. For a fixed x , the function $\alpha(\lambda)(x): Y \rightarrow Z$ is continuous because it is the composite

$$Y \cong \{x\} \times Y \subseteq X \times Y \xrightarrow{\lambda} Z.$$

Thus at least $\alpha(\lambda)$ is a function from X to $\text{Map}(Y, Z)$.

Proposition 2.9.7 *The function $\alpha(\lambda): X \rightarrow \text{Map}(Y, Z)$ is continuous.*

Proof. Consider $W_{K,U}$, where $K \subseteq Y$ is compact and $U \subseteq Z$ is open. If $x \in X$ is a point such that $\alpha(\lambda)(x) \in W_{K,U}$, then $\lambda(\{x\} \times K) \subseteq U$ or $\{x\} \times K \subseteq (\lambda)^{-1}(U)$. Since $\lambda^{-1}(U)$ is open and K is compact, there is an open set V in X such that

$$\{x\} \times K \subseteq V \times K \subseteq \lambda^{-1}(U).$$

That is

$$x \in V \subseteq (\alpha(\lambda))^{-1}(W_{K,U})$$

and so $\alpha(\lambda)$ is continuous. ♠

Thus the function $\alpha: \text{Map}(X \times Y, Z) \rightarrow \text{Map}(X, \text{Map}(Y, Z))$ is well-defined. Now we consider the pointed case. Let X , Y and Z be pointed spaces. Let $p: X \times Y \rightarrow X \wedge Y$ be the quotient map. Then we have the map

$$\text{id}_Z^p: \text{Map}_*(X \wedge Y, Z) \rightarrow \text{Map}_*(X \times Y, Z) \subseteq \text{Map}(X \times Y, Z).$$

Clearly α maps the image of id_Z^p into the subspace

$$\text{Map}_*(X, \text{Map}_*(Y, Z)) \subseteq \text{Map}(X, \text{Map}_*(Y, Z)) \subseteq \text{Map}(X, \text{Map}(Y, Z))$$

because if $\lambda: X \wedge Y \rightarrow Z$, then $\lambda \circ p: X \times Y \rightarrow Z$ has the property that

$$\lambda \circ p|_{X \vee Y}: X \vee Y \rightarrow Z$$

is the constant map and so $\alpha(\lambda)(x_0)(y) = \alpha(x)(y_0) = z_0$ for any x, y . Thus the association map $\alpha: \text{Map}(X \times Y, Z) \rightarrow \text{Map}(X, \text{Map}(Y, Z))$ induces the *reduced association map*

$$\bar{\alpha}: \text{Map}_*(X \wedge Y, Z) \rightarrow \text{Map}_*(X, \text{Map}_*(Y, Z))$$

with

$$[\bar{\alpha}(\lambda)(x)](y) = \lambda(x \wedge y)$$

for $x \in X, y \in Y$ and $\lambda: X \wedge Y \rightarrow Z$.

Proposition 2.9.8 *If X is Hausdorff, then the association map*

$$\alpha: \text{Map}(X \times Y, Z) \rightarrow \text{Map}(X, \text{Map}(Y, Z))$$

is continuous and therefore the reduced association map

$$\bar{\alpha}: Z^{X \wedge Y} \rightarrow (Z^Y)^X$$

is continuous.

Proof. By Lemma 2.9.5, it suffices to consider $\alpha^{-1}(W_{K,U})$, where $K \subseteq X$ is compact and $U \subseteq \text{Map}(Y, Z)$ is of the form $W_{L,V}$ for $L \subseteq Y$ compact and $V \subseteq Z$ open. Now

$$\alpha^{-1}(W_{K,U}) = \{\lambda | (\alpha(\lambda)(K) \subseteq W_{L,V})\} = \{\lambda | \lambda(K \times L) \subseteq V\} = W_{K \times L, V}.$$

Thus α is continuous. ♠

Theorem 2.9.9 (a) *For all spaces X, Y and Z , the functions*

$$\alpha: \text{Map}(X \times Y, Z) \rightarrow \text{Map}(X, \text{Map}(Y, Z)) \quad \text{and}$$

$$\bar{\alpha}: Z^{X \wedge Y} \rightarrow (Z^Y)^X$$

are one-to-one.

(b) *If Y is locally compact Hausdorff, then both α and $\bar{\alpha}$ are onto.*

(c) *If both X and Y are locally compact Hausdorff, then α is a homeomorphism.*

(d) If both X and Y are compact and Hausdorff, then $\bar{\alpha}$ is homeomorphism.

Proof. (a) We only show that $\bar{\alpha}$ is one-to-one. Let $\lambda, \mu: X \wedge Y \rightarrow Z$ such that $\alpha(\lambda) = \alpha(\mu)$. Then for any $x \in X$ and $y \in Y$, we have

$$\lambda(x \wedge y) = [\alpha(\lambda)(x)](y) = [\alpha(\mu)(x)](y) = \mu(x \wedge y),$$

so that $\lambda = \mu$.

(b) Let $\lambda: X \rightarrow \text{Map}(Y, Z)$ be a map. Let $\mu: X \times Y \rightarrow Z$ be the composite

$$X \times Y \xrightarrow{\lambda \times \text{id}_Y} \text{Map}(Y, Z) \times Y \xrightarrow{e} Z,$$

where e is the evaluation map. By Theorem 2.9.3, the evaluation e is continuous and so is μ . Clearly $\alpha(\mu) = \lambda$ and so α is onto. Now given a pointed map $\lambda': X \rightarrow \text{Map}_*(Y, Z)$, let $\mu': X \wedge Y \rightarrow Z$ be the composite

$$X \wedge Y \xrightarrow{\lambda' \wedge \text{id}_Y} \text{Map}_*(Y, Z) \wedge Y \xrightarrow{e} Z,$$

where e is the evaluation. Again by Theorem 2.9.3 e is continuous and so is μ' . Clearly $\bar{\alpha}(\mu') = \lambda'$ and so $\bar{\alpha}$ is onto.

(c) Certainly α is continuous, one-to-one and onto, so we have only to show that the inverse to α is continuous. Let θ be the composite

$$\theta: \text{Map}(X, \text{Map}(Y, Z)) \times X \times Y \xrightarrow{e \times \text{id}_Y} \text{Map}(Y, Z) \times Y \xrightarrow{e} Z,$$

where e are evaluations. By Theorem 2.9.3, θ is continuous. By Proposition 2.9.7, the function

$$\alpha(\theta): \text{Map}(X, \text{Map}(Y, Z)) \rightarrow \text{Map}(X \times Y, Z)$$

is continuous. Clearly $\alpha(\theta)$ is the inverse of the association map α .

(d) By Theorem 2.8.11, there is a homeomorphism

$$(Z^Y)^X \wedge (X \wedge Y) \cong ((Z^Y)^X \wedge X) \wedge Y.$$

Let ψ be the composite

$$(Z^Y)^X \wedge (X \wedge Y) \cong ((Z^Y)^X \wedge X) \wedge Y \xrightarrow{e \wedge \text{id}_Y} Z^Y \wedge Y \xrightarrow{e} Z.$$

Then ψ is continuous and

$$\bar{\alpha}(\psi): (Z^Y)^X \rightarrow Z^{X \wedge Y}$$

is the inverse to the reduced association $\bar{\alpha}$. ♠

Let X be a pointed space. The n -fold loop space $\Omega^n(X)$ of X is defined by

$$\Omega^n(X) = \text{Map}_*(S^n, X).$$

Exercise 2.9.2 Let X and Y be pointed spaces. Show that $\Omega^n(X \times Y) \cong \Omega^n(X) \times \Omega^n(Y)$ and $\Omega^{n+m}(X) \cong \Omega^m(\Omega^n(X))$.

2.10 Manifolds and Configuration Spaces

A Hausdorff space M is called an n -manifold if each point of M has a neighborhood homeomorphic to an open set in \mathbb{R}^n .

For example, \mathbb{R}^n and the n -sphere S^n is an n -manifold. A 2-dimensional manifold is called a *surface*. The objects traditionally called ‘surfaces in 3-space’ can be made into manifolds in a standard way. The compact surfaces have been classified as spheres or projective planes with various numbers of handles attached.

Exercise 2.10.1 Show that the real projective space $\mathbb{R}P^n$ is an n -manifold and the complex projective space $\mathbb{C}P^n$ is a $2n$ -manifold.

By the definition of manifold, the closed n -disk D^n is not an n -manifold because it has the ‘boundary’ S^{n-1} . D^n is an example of ‘manifolds with boundary’. We give the definition of manifold with boundary as follows.

A Hausdorff space M is called an n -manifold with boundary ($n \geq 1$) if each point in M has a neighborhood homeomorphic to an open set in the half space

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Manifold is one of models that we can do calculus ‘locally’. By means of calculus, we need local coordinate systems. Let $x \in M$. By the definition, there is a an open neighborhood $U(x)$ of x and a homeomorphism ϕ_x from $U(x)$ onto an open set in \mathbb{R}_+^n . The collection $\{(U(x), \phi_x) \mid x \in M\}$ has the property that 1) $\{U(x) \mid x \in M\}$ is an open cover and 2) ϕ_x is a homeomorphism from $U(x)$ onto an open set in \mathbb{R}_+^n . The subspace $\phi_x(U_x)$ in \mathbb{R}_+^n plays a role as a local coordinate system. The collection $\{(U(x), \phi_x) \mid x \in M\}$ is somewhat too large and we may like less local coordinate systems. This can be done as follows.

Let M be a space. A *chart* of M is a pair (U, ϕ) such that 1) U is an open set in M and 2) ϕ is a homeomorphism from U onto an open set in \mathbb{R}_+^n . An *atlas* for M means a collection of charts $\{(U_\alpha, \phi_\alpha) \mid \alpha \in J\}$ such that $\{U_\alpha \mid \alpha \in J\}$ is an open cover of M .

Proposition 2.10.1 *A Hausdorff space M is a manifold (with boundary) if and only if M has an atlas.*

Proof. Suppose that M is a manifold. Then the collection $\{(U(x), \phi_x) \mid x \in M\}$ is an atlas. Conversely suppose that M has an atlas. For any $x \in M$ there exists α such

that $x \in U_\alpha$ and so U_α is an open neighborhood of x that is homeomorphic to an open set in \mathbb{R}_+^n . Thus M is a manifold. ♠

We define a subset ∂M as follows: $x \in \partial M$ if there is a chart (U_α, ϕ_α) such that $x \in U_\alpha$ and $\phi_\alpha(x) \in \mathbb{R}^{n-1} = \{x \in \mathbb{R}^n | x_n = 0\}$. ∂M is called the boundary of M . For example the boundary of D^n is S^{n-1} .

Proposition 2.10.2 *Let M be a n -manifold with boundary. Then ∂M is an $(n-1)$ -manifold without boundary.*

Proof. Let $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ be an atlas for M . Let $J' \subseteq J$ be the set of indices such that $U_\alpha \cap \partial M \neq \emptyset$ if $\alpha \in J'$. Then Clearly

$$\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M} | \alpha \in J'\}$$

can be made into an atlas for ∂M . ♠

Definition 2.10.3 A Hausdorff space M is called a *differential manifold of class C^k* if there is an atlas of M

$$\{(U_\alpha, \phi_\alpha | \alpha \in J\}$$

such that

For any $\alpha, \beta \in J$, the composites

$$\phi_\alpha \circ \phi_\beta^{-1}: \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}_+^n$$

is differentiable of class C^k .

The atlas $\{(U_\alpha, \phi_\alpha | \alpha \in J\}$ is called a *differential atlas of class C^k* on M .

Two differential atlases of class C^k $\{(U_\alpha, \phi_\alpha) | \alpha \in I\}$ and $\{(V_\beta, \psi_\beta) | \beta \in J\}$ are called *equivalent* if

$$\{(U_\alpha, \phi_\alpha) | \alpha \in I\} \cup \{(V_\beta, \psi_\beta) | \beta \in J\}$$

is again a differential atlas of class C^k (this is an equivalence relation). A *differential structure of class C^k* on M is an equivalence class of differential atlases of class C^k on M . Thus a differential manifold of class C^k means a manifold with a differential structure of class C^k . A *smooth* manifold means a differential manifold of class C^∞ .

Note: A general manifold is also called *topological manifold*. Kervaire and Milnor [2] have shown that the topological sphere S^7 has 28 distinct oriented smooth structures.

Let M be a smooth manifold and let $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ be a C^∞ -atlas for M . For $\alpha, \beta \in J$, the function

$$\phi_\alpha \circ \phi_\beta^{-1}: \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}_+^n$$

is a smooth map from an open set in \mathbb{R}_+^n to an open set in \mathbb{R}_+^n . The Jacobian matrix

$$M_{\alpha\beta}(x) = \left(\frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})_i}{\partial x_j} \Big|_{\phi_\beta(x)} \right)$$

is invertible for any $x \in U_\alpha \cap U_\beta$. A smooth manifold M is called *orientable* if there is an C^∞ -atlas $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ for M such that the determinant of the Jacobian

$$\det(M_{\alpha\beta}(x)) > 0$$

for any $\alpha, \beta \in J$ and $x \in U_\alpha \cap U_\beta$. For example $\mathbb{R}P^n$ is orientable if and only if n is odd. On the other hand $\mathbb{C}P^n$ is orientable for any n .

Definition 2.10.4 let M and N be smooth manifolds of dimensions m and n respectively. A map $f: M \rightarrow N$ is called *smooth* if for some smooth atlases $\{(U_\alpha, \phi_\alpha) | \alpha \in I\}$ for M and $\{(V_\beta, \psi_\beta) | \beta \in J\}$ for N the functions

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} |_{\phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha)}: \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \rightarrow \mathbb{R}_+^n$$

are of class C^∞ .

Proposition 2.10.5 *If $f: M \rightarrow N$ is smooth with respect to atlases*

$$\{(U_\alpha, \phi_\alpha) | \alpha \in I\}, \quad \{(V_\beta, \psi_\beta) | \beta \in J\}$$

for M, N then it is smooth with respect to equivalent atlases

$$\{(U'_\delta, \theta_\delta) | \delta \in I'\}, \quad \{(V'_\gamma, \eta_\gamma) | \gamma \in J'\}$$

Proof. Since f is smooth with respect with the atlases

$$\{(U_\alpha, \phi_\alpha) | \alpha \in I\}, \quad \{(V_\beta, \psi_\beta) | \beta \in J\},$$

f is smooth with respect to the smooth atlases

$$\{(U_\alpha, \phi_\alpha) | \alpha \in I\} \cup \{(U'_\delta, \theta_\delta) | \delta \in I'\}, \quad \{(V_\beta, \psi_\beta) | \beta \in J\} \cup \{(V'_\gamma, \eta_\gamma) | \gamma \in J'\}$$

by look at the local coordinate systems. Thus f is smooth with respect to the atlases

$$\{(U'_\delta, \theta_\delta | \alpha \in I'\}, \{(V'_\gamma, \eta_\gamma | \beta \in J'\}. \spadesuit$$

Thus the definition of smooth maps between two smooth manifolds is independent of choice of atlas.

Let M be a m -manifold. The (*ordered*) *configuration space* $F(M, n)$ is defined by

$$F(M, n) = \{(x_1, \dots, x_n) \in M^n | x_i \neq x_j \text{ for } i \neq j\}.$$

In other words, the configuration space $F(M, n)$ is the subspace of the Cartesian product M^n by removing the ‘flat’ diagonals. The symmetric group Σ_n acts on $F(M, n)$ by permuting coordinates. The (*unordered*) *configuration space* $B(M, n)$ is the quotient of $F(M, n)$ by Σ_n , that is

$$B(M, n) = F(M, n)/\Sigma_n.$$

Clearly both $F(M, n)$ and $B(M, n)$ are mn -manifolds. configuration spaces are arisen from many areas in mathematics and physics. In geometry and physics, the diagonals play as singularities in many cases and so we have to remove them, then this gives the configuration space. In combinatorics, the homology of configuration spaces is related to ‘subspace arrangements’. The determination of the homology of $F(M, n)$ and $B(M, n)$ still remains open for general manifold M though it is known for many cases. The fundamental groups of configuration spaces are interesting as well. A typical example is that the fundamental group of $F(\mathbb{R}^2, n)$ is the pure braid group K_n and the fundamental group of $B(\mathbb{R}^2, n)$ is the Artin braid group B_n . The braid groups are important in group theory, low dimensional topology and mathematical physics. In homotopy theory, configuration spaces are used to construct various combinatorial models for mapping spaces. (As we have seen that mapping spaces are quite complicated, the construction means that we construct certain ‘simpler spaces’ that has the same homotopy groups and homology groups of a mapping space. So if one needs to know the homotopy groups and homology groups of a complicated mapping space, one may look at these simpler spaces.)