We show that ϕ' is continuous. Let C be a closed set in X. Then

$$\phi'^{-1}(C) \cap J_n(X) = \phi_n'^{-1}(C)$$

is a closed set in $J_n(X)$ for each n. Thus $\phi'(C)$ is closed because $J(X)$ has the weak topology with respect to $\{J_n(X)\}$. Since

$$\phi'|_{J_1(X)} = \phi_1' \colon J_1(X) = X \to X$$

is the identity map, the map ϕ' is a retraction and hence the result because $J(X)$ is a topological monoid. ♠

Note. If $*$ is non-degenerate, that is $* \to X$ is a cofibration, then X is an H-space with a homotopy identity if and only if X is an H-space with a strict identity. (See Proposition 3.3.1) Thus suppose that $*$ is non-degenerate, then X is an H-space if and only if X is a retract of a topological monoid.

Note. It is known that $J(X) \simeq \Omega\Sigma X$ if X is a path-connected CW-complex. (For this reason, $J(X)$ is known as a 'combinatorial model' for loop suspensions. For instance, $J(S^1) \simeq \Omega S^2$.) Thus suppose that X is a path-connected CW-complex with a non-degenerate base-point, then X is an H-space is and only if X is a retract of a loop space.

3.4 The fundamental Group

3.4.1 The fundamental Groupoid

Let λ and μ be two paths in X with $\lambda(1) = \mu(0)$. Then the product $\lambda * \mu$ is defined by

$$(\lambda * \mu)(t) = \begin{cases} \lambda(2t) & 0 \leq t \leq 1/2 \\ \mu(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Two paths λ and λ' are briefly said to be *homotopic*, denoted by $\lambda \simeq \lambda'$, if they are homotopic relative to $\partial I = \{0, 1\}$. Note that if $\lambda \simeq \lambda'$, then $\lambda(0) = \lambda'(0)$ and $\lambda(1) = \lambda'(1)$.

Lemma 3.4.1 *Let $\lambda_0, \lambda_1, \mu_0, \mu_1$ are paths in X with $\lambda_0(1) = \mu_0(0)$ and $\lambda_1(1) = \mu_1(0)$. If $\lambda_0 \simeq \lambda_1$ and $\mu_0 \simeq \mu_1$, then $\lambda_0 * \mu_0 \simeq \lambda_1 * \mu_1$.*

Proof. Let $F \colon lambda_0 \simeq \lambda_1$ and $G \colon \mu_0 \simeq \mu_1$ be the homotopies relative to ∂I. Then $H \colon I \times I \to X$ defined by

$$H(t, s) = \begin{cases} F(2t, s) & 0 \leq t \leq 1/2 \\ G(2t-1, s) & 1/2 \leq t \leq 1 \end{cases}$$

is a homotopy relative to ∂I between $\lambda_0 * \mu_0$ and $\lambda_1 * \mu_1$. ♠

Lemma 3.4.2 *Suppose that* $\lambda_0, \lambda_1, \lambda_2$ *are paths in* X *with* $\lambda_0(1) = \lambda_1(0)$ *and* $\lambda_1(1) = \lambda_2(0)$. *Then* $(\lambda_0 * \lambda_1) * \lambda_2 \simeq \lambda_0 * (\lambda_1 * \lambda_2)$.

Proof. The map $F: I \times I \to X$ defined by

$$F(t,s) = \begin{cases} \lambda_0((4t)/(1+s)) & 0 \leq t \leq (s+1)/4, \\ \lambda_1(4t-s-1) & (s+1)/4 \leq t \leq (s+2)/4, \\ \lambda_2((4t-s-2)/(2-s)) & (s+2)/4 \leq t \leq 1; \end{cases}$$

is a homotopy relative to ∂I between $(\lambda_0 * \lambda_1) * \lambda_2$ and $\lambda_0 * (\lambda_1 * \lambda_2)$.

For each $x \in X$, we define $\epsilon_x: I \to X$ as the constant path with $\epsilon_x(t) = x$ for any t.

Lemma 3.4.3 *Let* λ *be in path in* X *with* $\lambda(0) = x$ *and* $\lambda(1) = y$. *Then* $\epsilon_x * \lambda \simeq \lambda$ *and* $\lambda * \epsilon_y \simeq \lambda$.

Proof. The map $F: I \times I \to X$ defined by

$$F(t,s) = \begin{cases} x & 0 \leq t \leq (1-s)/t, \\ \lambda((2t-1+s)/(1+s)) & (1-s)/2 \leq t \leq 1; \end{cases}$$

is a homotopy relative to ∂I between $\epsilon_x * \lambda$ and λ. The map $G: I \times I \to X$ defined by

$$G(t,s) = \begin{cases} \lambda(\frac{2}{1+s}t) & 0 \leq t \leq \frac{1+s}{2} \\ y & \frac{1+s}{2} \leq t \leq 1. \end{cases}$$

is a homotopy relative to ∂I between $\lambda * \epsilon_y$ and λ. ♠

Given a path λ in X, the inverse λ^{-1} is defined by $\lambda^{-1}(t) = \lambda(1-t)$.

Lemma 3.4.4 *Let* λ *be a path in* X *with* $\lambda(0) = x$ *and* $\lambda(1) = y$. *Then* $\lambda * \lambda^{-1} \simeq \epsilon_x$ *and* $\lambda^{-1} * \lambda \simeq \epsilon_y$.

Proof. The map $F: I \times I \to X$ defined by

$$F(t,s) = \begin{cases} \lambda(2t(1-s)) & 0 \leq t \leq 1/2, \\ \lambda((2-2t)(1-s)) & 1/2 \leq t \leq 1; \end{cases}$$

is a homotopy relative to ∂I between $\lambda * \lambda^{-1}$ and ϵ_x. Similarly $\lambda^{-1} * \lambda \simeq \epsilon_y$. ♠

A category is called *small* if the class of objects is a set. A *groupoid* is a small category in which every morphism is an equivalence. Let X be a space. Let category $\mathcal{P}(X)$ is defined by:

the objects in $\mathcal{P}(X)$ are points in X and morphisms from x to y are path classes from x to y. The composite operation is defined by $[\mu] \circ [\lambda] = [\lambda * \mu]$ for a path λ from x to y and a path μ from y to z.

By the lemmas above, we have

Theorem 3.4.5 *Let X be a space. Then $\mathcal{P}(X)$ is a groupoid.*

3.4.2 Change of Base

Let X be a space with $x \in X$. Consider x is the basepoint of X. Then $\pi_1(X, x) = \pi_1(X)$ is called the *fundamental group* of X with base point x. Recall that $\pi_1(X, x)$ is a group, where the multiplication is given by the path multiplication. Note that the fundamental group depends on the choice of the base point x.

Theorem 3.4.6 *Let $x, y \in X$. If there is a path in X from x to y, then the groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic.*

Proof. Let λ be a path from x to y, that is $\lambda(0) = x$ and $\lambda(1) = y$. Define a function

$$\chi_\lambda \colon \pi_1(X, x) \to \pi_1(X, y)$$

by

$$\chi_\lambda([\mu]) = [\lambda^{-1} * \mu * \lambda].$$

This is a homomorphism of groups because

$$\chi_\lambda([\mu][\mu']) = [\lambda^{-1} * \mu * \mu' * \lambda] = [\lambda^{-1} * \mu * \lambda * \lambda^{-1} * \mu' * \lambda]$$

$$= [\lambda^{-1} * \mu\lambda][\lambda^{-1} * \mu * \lambda] = \chi_\lambda([\mu])\chi_\lambda([\mu']).$$

λ^{-1} is path from y to x and so

$$\chi_{\lambda^{-1}} \colon \pi_1(X, y) \to \pi_1(X, x).$$

For $\mu \in \pi_1(X, x)$, we have

$$\chi_{\lambda^{-1}} \circ \chi_\lambda([\mu]) == [\lambda * \lambda^{-1}\mu * \lambda * \lambda^{-1}] = [\mu]$$

and so $\chi_{\lambda^{-1}} \circ \chi_\lambda = \text{id}$. Similarly $\chi_\lambda \circ \chi_{\lambda^{-1}} = \text{id}$. Thus χ_λ is an isomorphism of groups. ♠

Let $f\colon X \to Y$ be a map. Then f induces a homomorphism of groups

$$f_*\colon \pi_1(X,x) = [S^1, X] \to \pi_1(Y, f(x))[S^1, Y].$$

If $f \simeq g \operatorname{rel} x$, then

$$f_* = g_*\colon \pi_1(X,x) \to \pi_1(Y,y),$$

where $y = f(x) = g(x)$. If $X \simeq Y$ relative the base-point, then $\pi_1(X) \cong \pi_1(Y)$.

Exercise 3.4.1 *Prove that if there is a path in X from x_0 to x_1, then $\pi_n(X,x_0)$ and $\pi_n(X,x_1)$ are isomorphic.*

3.4.3 The fundamental Group of a Circle

The map $\mathrm{e}\colon \mathbb{R} \to S^1$ is defined by

$$\mathrm{e}(t) = \exp^{2\pi i t}.$$

Then e is continuous, $\mathrm{e}(t_1 + t_2) = \mathrm{e}(t_1)\,\mathrm{e}(t_2)$ and $\mathrm{e}(t_1) = \mathrm{e}(t_2)$ if and only if $t_1 - t_2$ is an integer. It follows that $\mathrm{e}\,|_{(-1/2,1/2)}$ is a homeomorphism of the open interval $(-1/2, 1/2)$ onto $S^1 \setminus \{\exp(\pi i)\}$. Let

$$\log\colon S^1 \setminus \{\exp(\pi i)\} \to (-1/2, 1/2)$$

be the inverse of $\mathrm{e}\,|_{(-1/2,1/2)}$.

A subset $X \subseteq \mathbb{R}^n$ is called *starlike* from a point x_0 if, whenever $x \in X$, the closed segment $[x_0, x]$ from x_0 to x lies in X.

Lemma 3.4.7 *Let X be compact and starlike from $x_0 \in X$. Given any continuous map $f\colon X \to S^1$ and any $t_0 \in \mathbb{R}$ such that $\mathrm{e}(t_0) = f(x_0)$, there exists a continuous map $\tilde{f}\colon X \to \mathbb{R}$ such that $\tilde{f}(x_0) = t_0$ and $\mathrm{e} \circ \tilde{f}(x) = f(x)$ for all $x \in X$.*

Proof. Clearly we can translate X so that it is starlike from the origin; hence there is no loss of generality in assuming $x_0 = 0$. Since X is compact, f is uniformly continuous and there exists $\epsilon > 0$ such that if $\|x - x'\| < \epsilon$, then $\|f(x) - f(x')\| < 2$ [that is, $f(x)$ and $f(x')$ are not antipodes in S^1]. Since X is bounded, there exists a positive integer n such that $\|x\|/n < \epsilon$ for all $x \in X$. Then for each $0 \le j < n$ and all $x \in X$

$$\|\frac{(j+1)x}{n} - \frac{jx}{n}\| = \frac{\|x\|}{n} < \epsilon$$

and so

$$\|f\left(\frac{(j+1)x}{n}\right)-f\left(\frac{jx}{n}\right)\|<2.$$

It follows that the quotient $f((j+1)x/n)/f(jx/n)$ is a point of $S^1\setminus\{\exp(\pi i)\}$. Let $g_j\colon X\to S^1\setminus\{\exp(\pi i)\}$ for $0\le j<n$ be the map defined by

$$g_j(x)=\frac{f((j+1)x/n)}{f(jx/n)}.$$

Then for all $x\in X$, we see that

$$f(x)=f(0)g_0(x)g_1(x)\cdots g_{n-1}(x).$$

We define $\tilde{f}\colon X\to\mathbb{R}$ by

$$\tilde{f}(x)=t_0+\log(g_0(x))+\log(g_1(x))+\cdots+\log(g_{n-1}(x)).$$

Since f' is the sum of $n+1$ continuous functions from X to $\mathbb{R}$, it is continuous. Clearly $\tilde{f}(0)=t_0$ and $\mathrm{e}\circ\tilde{f}=f$. ♠

Lemma 3.4.8 *Let X be a connected space and let $\tilde{f},\tilde{g}\colon X\to\mathbb{R}$ be maps such that $\mathrm{e}\circ\tilde{f}=\mathrm{e}\circ\tilde{g}$ and $\tilde{f}(x_0)=\tilde{g}(x_0)$ for some $x_0\in X$. Then $\tilde{f}=\tilde{g}$.*

Proof. Let $h=\tilde{f}-\tilde{g}\colon X\to\mathbb{R}$. Since $\mathrm{e}\circ\tilde{f}=\mathrm{e}\circ\tilde{g}$, $\mathrm{e}\circ h$ is the constant map of X to $1\in S^1$. Thus h is a continuous map from X to $\mathbb{R}$, taking only integral values. Because X is connected, h is constant, and since $h(x_0)=0$, $h(x)=0$ for all $x\in X$. ♠

Let $\alpha\colon I\to S^1$ be a closed path at 1. Because I is starlike from 0 and $\alpha(0)=1=\mathrm{e}(0)$, it follows from Lemmas 3.4.7 and 3.4.8 there exists a unique lifting $\tilde{\alpha}\colon I\to\mathbb{R}$ such that $\tilde{\alpha}(0)=0$ and $\mathrm{e}\circ\tilde{\alpha}=\alpha$. Because $\mathrm{e}(\tilde{\alpha}(1))=\alpha(1)=1$, it follows that $\tilde{\alpha}(1)$ is an integer. We define the degree of α by

$$\deg(\alpha)=\tilde{\alpha}(1).$$

Lemma 3.4.9 *Let α and β be homotopic closed paths in S^1 at 1. Then* $\deg(\alpha)=\deg(\beta)$.

Proof. Let $F\colon I\times I\to S^1$ be a homotopy relative to ∂I from α to β. Because $I\times I$ is a starlike set of $\mathbb{R}^2$ from $(0,0)$, it follows that there is a (unique) lifting $\tilde{F}\colon I\times I\to\mathbb{R}$ such that $\tilde{F}(0,0)=0$ and $\mathrm{e}\circ\tilde{F}=F$. Since F is a homotopy relative to ∂I, $F(0,t)=F(1,t)=1$ for all $t\in I$. Thus $\tilde{F}(0,t)$ and $\tilde{F}(1,t)$ take on only integral values for all $t \in I$. It follows that $\tilde{F}(0,t)$ must be constant and $\tilde{F}(1,t)$ must be constant. Because $\tilde{F}(0,0) = 0$, $\tilde{F}(0,t) = 0$ for all t. Let $\tilde{\alpha}, \tilde{\beta} \colon I \to \mathbb{R}$ be the maps defined by $\tilde{\alpha}(t) = \tilde{F}(t,0)$ and $\tilde{\beta}(t) = \tilde{F}(t,1)$. Then $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$, $\mathrm{e} \circ \tilde{\alpha} = \alpha$ and $\mathrm{e} \circ \tilde{\beta} = \beta$. Thus

$$\deg(\alpha) = \tilde{\alpha}(1) = \tilde{F}(1,0) = \tilde{F}(1,t) = \tilde{F}(1,1) = \tilde{\beta}(1) = \deg(\beta). \spadesuit$$

It follows that there is a well-defined function deg from $\pi_1(S^1, 1)$ to $\mathbb{Z}$ defined by

$$\deg([\alpha]) = \deg(\alpha).$$

Theorem 3.4.10 *The function* deg *is an isomorphism of groups*

$$\deg \colon \pi_1(S^1, 1) \cong \mathbb{Z}.$$

Proof. To prove that deg is a homomorphism, let α and β be two closed paths in S^1 at 1 and let $\alpha\beta$ be the closed path which is their pointwise product in the group multiplication of S^1. We know from Theorem 3.3.13 that $[\alpha] * [\beta] = [\alpha\beta]$. Let $\tilde{\alpha}, \tilde{\beta} \colon I \to \mathbb{R}$ be such that $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$, $\mathrm{e} \circ \tilde{\alpha} = \alpha$ and $\mathrm{e} \circ \tilde{\beta} = \beta$. Let $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta} \colon I \to \mathbb{R}$. Then $\tilde{\gamma}(0) = 0$ and $\mathrm{e}(\tilde{\gamma}) = \alpha\beta$. Thus

$$\deg([\alpha] * [\beta]) = \deg([\alpha\beta]) = \tilde{\gamma}(1) = \tilde{\alpha}(1) + \tilde{\beta}(1) = \deg([\alpha]) + \deg([\beta]).$$

The map deg is an epimorphism: For any integer n, let $\tilde{\alpha} \colon I \to \mathbb{R}$ be the path defined by $\tilde{\alpha}(t) = tn$ and let $\alpha = \mathrm{e} \circ \tilde{\alpha} \colon I \to S^1$. Then clearly $\deg([\alpha]) = n$.

The map deg is a monomorphism: If $\deg([\alpha]) = 0$, then there is a path $\tilde{\alpha} \colon I \to \mathbb{R}$ with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = 0$ and $\mathrm{e} \circ \tilde{\alpha} = \alpha$. Since $\mathbb{R}$ is contractible, $\tilde{\alpha} \simeq \epsilon_0$ and

$$\alpha = \mathrm{e} \circ \tilde{\alpha} \simeq \mathrm{e} \circ \epsilon_0 = \epsilon_1. \spadesuit$$

Exercise 3.4.2 *Show that the map $f \colon S^1 \to S^1$, $z \to z^n$ is of degree n.*

Corollary 3.4.11 *The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.*

Theorem 3.4.12 (The Fundamental Theorem of Algebra) *Every non-constant complex polynomial has a root.*

Proof. We may assume without loss of generality that our polynomial has the form

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

with $n \geq 1$. Assume that p has no zero.

Let S_r be the circle $|z| = r$ of radius r. Choose $r >> 0$ such that

$$r^n > |a_1|r^{n-1} + |a_2|r^{n-2}| + \cdots + |a_{n-1}|r + |a_n|.$$

Let $F\colon S_r \times I \to \mathbb{C}$ be the map defined by

$$F(z,t) = z^n + t(a_1 z^{n-1} + \cdots + a_n).$$

Since

$$|F(z,t)| \geq |z|^n - t(|a_1||z|^{n-1} + \cdots + |a_n|) > 0$$

for $|z| = r$ and $0 \leq t \leq 1$, the image of F lies in $\mathbb{C} \setminus \{0\}$. Let

$$G\colon S^1 \times I \to S^1$$

be the composite

$$S^1 \xrightarrow{rz} S_r \xrightarrow{\frac{F(z,t)}{F(r,t)}} \mathbb{C} \setminus \{0\} \xrightarrow{\frac{z}{|z|}} S^1.$$

Then $G(1,t) = (F(r,t)/F(r,t))/|F(r,t)/F(r,t)| = 1$ for $t \in I$, $G(z,0) = z^n$ and

$$G(z,1) = \frac{p(rz)}{p(r)} \frac{|p(r)|}{|p(rz)|}.$$

Thus $f(z) = (|p(r)|/(p(r)|p(rz)|))p(rz) \simeq z^n$ is of degree n.

Let $H\colon S^1 \times I \to S^1$ be the map defined by

$$H(z,t) = \frac{p(rzt}{p(rt)} \frac{|p(rt)|}{|p(rzt)|},$$

where H is well-defined (and so it is continuous) because $p(z)$ is never zero. Then $H(1,t) = 1$ for all t, $H(z,0) = 1$ and $H(z,1) = f(z)$. It follows that $f(z)$ is of degree 0, which is a contradiction (unless $n = 0$).

Theorem 3.4.13 (Brouwer Fixed Point Theorem) *Any continuous map $f\colon D^2 \to D^2$ has a fixed point, that is a point x such that $f(x) = x$.*

Proof. Suppose that $x \neq f(x)$ for all $x \in D^2$. Then we may define a map $\phi\colon D^2 \to S^1$ by setting $\phi(x)$ to be the point on S^1 obtained from the intersection of the line segment from $f(x)$ to x extended to meet S^1. Let $i\colon S^1 \to D^2$ be the inclusion. Then $\phi \circ i = \mathrm{id}_{S^1}$. Thus there is a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} = \pi_1(S^1) \arrow[r, equal] \arrow[d, "i_*"] & \mathbb{Z} = \pi_1(S^1) \\
0 = \pi_1(D^2) \arrow[r, equal] & 0 = \pi_1(D^2), \arrow[u, "\phi_*"']
\end{tikzcd}$$

which is impossible. This contradiction proves the result. ♠

Exercise 3.4.3 Show that $\pi_n(S^1) = 0$ for $n > 1$. (Hint: Let $q\colon I^n \to S^n = I^n/\partial I^n$ the pinch map. Let $f\colon S^n \to S^1$ be any map. Consider $f \circ q\colon I^n \to S^1$. Since I^n is starlike, there is a unique lifting $\alpha\colon I^n \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\mathrm{e} \circ \alpha = f \circ q$. Since

$$\mathrm{e} \circ \alpha(x) = f \circ q(x) = f(*) = 1$$

for $x \in \partial I^n$, $\mathrm{e} \circ \alpha|_{\partial I^n}$ is the constant map and so $\alpha|_{\partial I^n}$ is a continuous map from ∂I^n to integers. It follows that $\alpha|_{\partial I^n}$ is a constant map because $\partial I^n \cong S^{n-1}$ is path-connected when $n > 1$. Since $\alpha(0) = 0$, $\alpha(x) = 0$ for $x \in \partial I^n$ and so α induces a map $\bar{\alpha}\colon S^n = I^n/\partial I^n \to \mathbb{R}$. Since $\mathrm{e} \circ \alpha = f \circ q$, we have $\mathrm{e} \circ \bar{\alpha} = f$. Since $\mathbb{R}$ is contractible, $\bar{\alpha} \simeq \epsilon_0$ and so

$$f = \mathrm{e} \circ \alpha \simeq \mathrm{e} \circ \epsilon_0 = \epsilon.$$

This show that any map from S^n to S^1 is null homotopic and so $\pi_n(S^1) = 0$.)

3.4.4 Simply Connected Spaces

A space X is said to be *n-connected* for $n \geq 0$ if every continuous map $f\colon S^k \to X$ for $k \leq n$ has a continuous extension over E^{k+1}. A 1-connected space is also said to *simply connected*. Note that if $0 \leq m \leq n$, an n-connected space is m-connected. It follows from Theorem 3.1.16 that a space X is n-connected if and only if it is path-connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$. By Exercise 3.4.1, X is n-connected if and only if it is path-connected and $\pi_k(X, x_0) = 0$ for $1 \leq k \leq n$ and any particular choice of base point x_0. Note that X is 0-connected if and only if X is path-connected. By Exercise 3.1.5, we have

Lemma 3.4.14 *A contractible space is n-connected for every $n \geq 0$.*

Exercise 3.4.4 Let λ and μ be paths in X from x to y. Suppose that X is simply connected. Then $\lambda \simeq \mu$.

Lemma 3.4.15 *Suppose that $X = U \cup V$ with U, V open and simply connected and $U \cap V$ non-empty and path connected. Then X is simply connected.*

Proof. Let f be any path in X. Then $f^{-1}(U)$ is an open set of I and so $f^{-1}(U)$ is a disjoint union of open intervals. Let

$$f^{-1}(U) = \bigcup_{\alpha}(a_\alpha, b_\alpha)$$

be a disjoint union of open intervals (a_α, b_α). Since $f^{-1}(V)$ is open in I,

$$f^{-1}(V) = \bigcup_{\beta}(c_\beta, d_\beta).$$

Since

$$I = \bigcup_{\alpha,\beta}(a_\alpha, b_\alpha) \cup (c_\beta, d_\beta)$$

and I is compact, there exists a finite subcover

$$I = \bigcup_{i=1}^{m}(a_i, b_i) \cup \bigcup_{j=1}^{n}(c_j, d_j).$$

It follows that there are finite numbers

$$t_1 = 0 < t_2 < \cdots < t_q = 1$$

such that $[t_s, t_{s+1}]$ is either contained in (a_i, b_i) for some i or in (c_j, d_j) for some j. Let

$$f_s(t) = f(t_s + t(t_{s+1} - t_s)).$$

Then f_s is a path that starts with $f(t_s)$ and ends with $f(t_{s+1})$. If $[t_s, t_{s+1}] \subseteq (a_i, b_i)$ for some i, then $f_s(I) = f([t_s, t_{s+1}]) \subseteq U$, that is f_s is a path in U. Otherwise, $[t_s, t_{s+1}] \subseteq (c_j, d_j)$ for some j and f_s is a path in V. It follows that

$$f = f_1 * f_2 * \cdots * f_q,$$

where f_s is either in U or V and so

$$[f] = [f_1][f_2] \cdots [f_q].$$

We show by induction that

> *If f is a loop with $f(0) = f(1) \in U \cap V$ such that $[f] = [f_1][f_2] \cdots [f_q]$ with f_j is either a path in U or a path in V, then $[f] = 0$.*

The assertion will follow from this statement.

If $q = 1$, then $[f] = [f_1]$ and so f_1 must be a loop. If f_1 is a loop in U, then $[f_1] = 0$ because U is simply connected and so $[f] = 0$. Otherwise f_1 is a loop in V and $[f] = [f_1] = 0$ because V is simply connected. Assume that the statement holds for $< q$. Let $[f] = [f_1] \cdots [f_q]$. We may assume that f_1 is a path in U without loss of generality. Let $i \geq 1$ be the largest integer such that f_j is a path in U for $j \leq i$. Then $f_1 * f_2 * \cdots * f_i$ is a path in U and f_{i+1} is a path in V. It follows that

$$f_i(1) = f_{i+1}(0) \in U \cap V.$$

Since $U \cap V$ is path connected, there is a path λ in $U \cap V$ from $f(0)$ to $f_i(1)$. Since U is simply connected, and $f_1 * \cdots * f_i$ and λ are paths in U from $f(0)$ to $f_i(1)$, we have

$$[f_1] \cdots [f_i] = [\lambda]$$

and so

$$[f] = [\lambda][f_{i+1}][f_{i+2}] \cdots [f_q] = [\lambda * f_{i+1}][f_{i+2}] \cdots [f_q] = 0$$

by induction, where $\lambda * f_{i+1}$ is a path in V. By induction. ♠

Corollary 3.4.16 *S^n is simply connected for $n \geq 2$.*

Exercise 3.4.5 Let X be a space. The *unreduced suspension* $\Sigma^u X$ is the quotient space of $I \times X$ obtained by identifying $0 \times X$ to a point and $1 \times X$ to a (different) point. Suppose that X is path-connected. Show that $\Sigma^u X$ is simply connected.

Note: $\Sigma X = \Sigma^u X / I \times *$. If $I \times * \to \Sigma^u X$ is a cofibration (this is true if $* \to X$ is a cofibration), then $\Sigma^u X \simeq \Sigma X$. Thus if $* \to X$ is a cofibration and X is path-connected, then ΣX is simply connected.

3.5 The Seifert-Van Kampen Theorem

In this section, we provide a useful theorem for calculations of fundamental groups.

3.5.1 Free Groups and Free Products of groups

Let X be a set. The *free group* $F(X)$ generated by X is a group that satisfies the following universal property:

1) $X \subseteq F(X)$ is a subset.

2) Let G be any group and let $f\colon X \to G$ be any function. There exists a unique homomorphism of groups $\tilde{f}\colon F(X) \to G$ such that $\tilde{f}|_X$.

It is known that for any X $F(X)$ exists and unique up to isomorphism. There is an explicit construction of the free group $F(X)$ in terms of words:

$$w = x_1^{\epsilon_1} \cdots x_k^{\epsilon_k},$$

where $x_j \in X$ and $\epsilon_j = \pm 1$. For instance, if $X = \{x_1, \cdots, x_n\}$, the words on X are given by

$$x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$$

for $k \leq 0$, $\epsilon_j = \pm 1$ and $1 \leq i_j \leq n$. A word w is called *reduced* if for each $1 \leq j \leq k$ $x_j \neq x_{j+1}$ or $x_j = x_{j+1}$ with $\epsilon_j \neq -\epsilon_{j+1}$. As a set $F(X)$ is given by all reduced words and the multiplication on $F(X)$ is given by the formal product of words, where we use the rule:

$$x_i^{-1} x_i = x_i x_i^{-1} = 1$$

for each i. For example,

$$(x_1 x_2^{-1} x_3) \cdot (x_3^{-1} x_1) = x_1 x_2^{-1} x_1.$$

Clearly $F(X)$ is NOT a commutative group if X has more than one element because $x_1 x_2$ and $x_2 x_1$ are different words in $F(X)$ for $x_1, x_2 \in X$.

Definition 3.5.1 Let $f\colon H \to G$ and $g\colon H \to K$ be homomorphisms of groups. The *push-out* $G \coprod_H K$ is a group that satisfies the following universal properties:

1) There are homomorphisms of groups $\phi\colon G \to G \coprod_H K$ and $\psi\colon K \to G \coprod_H K$

such that $\phi \circ f = \psi \circ g$, that is the diagram

$$\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, "g"] & G \arrow[d, "\phi"] \\
K \arrow[r, "\psi"] & G \coprod_H K
\end{tikzcd}$$

,

commutes;

2) Let Γ be any group and let $\phi' \colon G \to \Gamma$ and $\psi' \colon K \to \Gamma$ be homomorphisms with $\phi' \circ f = \psi' \circ g$. Then there is a unique homomorphism $\theta \colon G \coprod_H K \to \Gamma$ such that $\phi' = \theta \circ \phi$ and $\psi' = \theta \circ \psi$.

When H is the trivial group, $G \coprod K = G \coprod_{\{1\}} K$ is called the *free product* of G and K. It is known in group theory that the push-out (so-called free product with amalgamation in group theory) always exists. The universal property show that $G \coprod_H K$ must be unique up to isomorphism if it exists. The combinatorial construction $G \coprod_H K$ can be given as follows:

First we construct the free product $G \coprod K$ can be given by the words

$$w = \alpha_1 \cdots \alpha_k,$$

where $\alpha_j \in G$ or K for each j. w is reduced if each $\alpha_j \neq 1$ and α_j and α_{j+1} are not lie in the same group. The product of two reduced words is the reduced words obtained from the formal product of them. For instance, let $\alpha_1, \alpha_2 \in G$ and $\beta_1 \in K$. Then

$$(\alpha_1 \beta_1 \alpha_2)(\alpha_2^{-1} \beta_1^{-1} \alpha_2) = \alpha_1 \alpha_2 \in G.$$

The push-out $G \coprod_H K$ is the quotient group of $G \coprod K$ by the normal subgroup generated by

$$f(h) g(h)^{-1}$$

for $h \in H$. We can check that this construction satisfies the universal property: The homomorphisms $\phi \colon G \to G \coprod_H K$ and $\psi \colon K \to G \coprod_H K$ are canonical map given by $\phi(g)$ is word represented by g and $\psi(k)$ is the word represented by k for $g \in G$ and $k \in K$. By the relation above $\phi \circ f = \psi \circ g$ in the group $\coprod_H K$ (NOT $G \coprod K$).

Assume that $\phi' \colon G \to \Gamma$ and $\psi' \colon K \to \Gamma$ be homomorphisms with $\phi' \circ f = \psi' \circ g$. First there is a unique homomorphism $\theta' \colon G \coprod K \to \Gamma$ such that $\theta'(g) = \phi'(g)$ and $\theta'(k) = \psi'(k)$. Since $\phi' \circ f = \psi' \circ g$, we have that

$$\theta'(f(h)g(h)^{-1}) = 1$$

for $h \in H$ and so θ' induces a unique homomorphism $\theta \colon G \coprod_H K \to \Gamma$ with the desired property.

Example 3.5.2 If K is the trivial group, then $G \coprod K = G$ and so $G \coprod_H K$ is the quotient group of G by the normal subgroup generated by the image of $f \colon H \to G$.

$\mathbb{Z} \coprod \mathbb{Z} = F(x_1, x_2)$ is a free group generated by two generators. In general, the n-fold free product of $\mathbb{Z}$ is a free group of rank n, that is n free generators.

$\mathbb{Z}/m \coprod \mathbb{Z}/n$ is the quotient group of $F(x_1, x_2)$ by the relations:

$$x_1^m = 1, x_2^n = 1.$$

3.5.2 The Seifert-Van Kampen Theorem

Theorem 3.5.3 *Let X be a space. Suppose that $X = U_1 \cup U_2$ such that U_1, U_2 are open and $U_1 \cap U_2$ is non-empty and path connected. Let $x_0 \in U_1 \cap U_2$ be a base-point of X. Then*

$$\pi_1(X, x_0) = \pi_1(U_1, x_0) \coprod_{\pi_1(U_1 \cap U_2, x_0)} \pi_1(U_2, x_0).$$

Sketch of Proof. Let $j^1 \colon U_1 \to X$, $j^2 \colon U_2 \to X$, $i^1 \colon U_1 \cap U_2 \to U_1$ and $i^2 \colon U_1 \cap U_2 \to U_2$ be inclusions. Since $j^1 \circ i^1 = j^2 \circ i^2$, the homomorphisms $j^1_* \colon \pi_1(U_1) \to \pi_1(X)$ and $j^2_* \colon \pi_1(U_2) \to \pi_1(X)$ induces a homomorphism

$$\theta \colon \pi_1(U_1) \coprod_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \to \pi_1(X).$$

Let $\lambda \colon S^1 \to X$ be a loop in X. By the proof of Lemma 3.4.15, we have

$$[\lambda] = [\lambda_1][\lambda_2] \cdots [\lambda_k],$$

where λ_j is a path either in U_1 or U_2. We may assume that λ_1 is a path in U_1. Let i be the largest number such that $\lambda_1, \cdots, \lambda_i$ are paths in U_1. Then $\lambda_i(1) = \lambda_{i+1}(0) \in$

$U_1 \cap U_2$. Since $U_1 \cap U_2$ is path connected, there is a path μ in $U_1 \cap U_2$ from $\lambda(0)$ to $\lambda_i(1) = \lambda_{i+1}(0)$. Then

$$[\lambda] = [\lambda_1 * \cdots * \lambda_i * \mu^{-1}][\mu\lambda_{i+1}][\lambda_{i+2}] \cdots [\lambda_k].$$

Now $\lambda_1 * \cdots * \lambda_i * \mu^{-1}$ is a loop in U_1 and $\mu * \lambda_{i+1}$ is a path in U_2. By repeating this step (one can do this by induction), finally one can written down $[\lambda]$ as a product of elements from $\pi_1(U_1)$ or $\pi_1(U_2)$ and so θ is an epimorphism.

It is more complicated to show that θ is a monomorphism. So we omit this part of proof. ♠

3.5.3 Calculations of the fundamental Group

By Using the Seifert-van Kampen theorem, we can compute the fundamental groups of a lot of spaces.

Example 3.5.4 $\pi_1(S^1 \vee S^1) = F(x_1, x_2)$. *In general,* $\pi_1(\vee^n S^1) = F(x_1, \cdots, x_n)$.

Proof. Let x be an element in S^1 different from the base point. Let $U = S^1 \vee (S^1 \setminus \{x\})$ and $V = (S^1 \setminus \{x\}) \vee S^1$. Then U and V are open sets in $S^1 \vee S^1$. Since $U \simeq S^1$ and $V \simeq S^1$, we have $\pi_1(U) = \mathbb{Z}$ and $\pi_1(V) = \mathbb{Z}$. Now $U \cap V = (S^1 \setminus \{x\}) \vee (S^1 \setminus \{x\})$ is contractible, $\pi_1(U \cap V) = \{1\}$. By the Seifert-van Kampen theorem, we have

$$\pi_1(S^1 \vee S^1) = \mathbb{Z} \coprod_{\{1\}} \mathbb{Z} = F(x_1, x_2).$$

By induction, one can show that $\pi_1(\vee^n S^1) = F(x_1, \cdots, x_n)$. ♠
Note: By this example, we know that $\vee^n S^1$ is NOT an H-space if $n > 1$.

Example 3.5.5 $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$ *and* $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ *for* $n \geq 2$.

Proof. Clearly $\mathbb{R}P^1 \cong S^1$ and so $\pi_1 \mathbb{R}P^1 = \mathbb{Z}$. Now we compute $\pi_1 \mathbb{R}P^2$.

Recall that $\mathbb{R}P^2/\mathbb{R}P^1 \cong S^2$. Let $x \in \mathbb{R}P^2 \setminus \mathbb{R}P^1$ and let $U = \mathbb{R}P^2 \setminus \{x\}$. Then U is homotopy equivalent to $\mathbb{R}P^1$ and so $\pi_1(U) = \mathbb{Z}$. Let V be an open neighborhood of x that is homeomorphic to the open disk B^2 and is disjoint from $\mathbb{R}P^1$. Then $\pi_1 V = 0$. Clearly $U \cap V \simeq S^1$, $\pi_1(U \cap V) = \mathbb{Z}$. Let $j \colon U \cap V \to U$ be the inclusion. Then $j_* \colon \pi_1(U \cap V) \to \pi_1(U)$ is multiple by 2. Thus by the Seifert-van Kampen theorem $\pi_1(\mathbb{R}P^2) = \pi_1(U \cup V) = \mathbb{Z}/2\mathbb{Z}$.

Now we show that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ by induction. Assume that $\pi_1(\mathbb{R}P^{n-1}) = \mathbb{Z}/2$ with $n \geq 3$. Let $x \in \mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$. Let $U = \mathbb{R}p^n \setminus \{x\}$. Then $U \simeq \mathbb{R}P^{n-1}$ and

$\pi_1(U) = \mathbb{Z}/2$. Let V be a small neighborhood of x with $V \cong B^n$. Then $\pi_1(V) = 0$. Clearly $U \cap V \simeq S^{n-1}$. Since $n \geq 3$, S^{n-1} is simply connected and so $\pi_1(U \cap V) = 0$. It follows that $\pi_1(\mathbb{R}P^n) = \pi_1(U \cup V) = \mathbb{Z}/2$. ♠

Exercise 3.5.1 Show that $\mathbb{C}P^n$ is simply connected for each $n \geq 1$.

Note: By looking at fundamental groups, we already know that any $\mathbb{R}P^m$ is NOT homeomorphic to $\mathbb{C}P^n$.

Exercise 3.5.2 Let $T_g = T\#T\#\cdots\#T$ be the g-fold connected sum of the torus T. Show that $\pi_1(T_g)$ is the quotient group of the free group $F(c_1, d_1, c_2, d_2, \cdots, c_g, d_g)$ by the one relation:

$$c_1 d_1 c_1^{-1} d_1^{-1} c_2 d_2 c_2^{-1} d_2^{-1} \cdots c_g d_g c_g^{-1} d_g^{-1} = 1.$$

3.5.4 Groups and Spaces

Let X be a space. The *unreduced cone* $CX = I \times X / 1 \times X$. Clearly the cone CX is contractible for any X. There is a relation between groups and so-called 2-complexes.

Lemma 3.5.6 *Let $\phi\colon F(x_1, \cdots, x_m) \to F(y_1, \cdots, y_n)$ be a homomorphism. Then there is a (continuous) map $f\colon \vee^m S^1 \to \vee^n S^1$ such that*

$$f_* = \phi\colon \pi_1(\bigvee^m S^1) = F(x_1, \cdots, x_m) \to \pi_1(\bigvee^n S^1) = F(y_1, \cdots y_n).$$

Proof. The homomorphism ϕ is uniquely determined by the elements $\phi(x_1), \cdots, \phi(x_m)$ in $F(y_1, \cdots, y_n)$. Since $\pi_1(\vee^n S^1) = F(y_1, \cdots, y_n)$, there are maps

$$f_1, \cdots, f_m\colon S^1 \to \bigvee^n S^1$$

such that $[f_j] = (f_j)_*([\mathrm{id}]) = \phi(x_j)$. Let $f\colon \vee^m S^1 \to \vee^n S^1$ be the map induced by $f_1, \cdots, f_m$. The $f_* = \phi$. ♠

Let G be a group with generators $x_1, \cdots, x_k$ and relations $R_1, \cdots, R_q$, where each R_j is a word in the free group $F(x_1, \cdots, x_k)$. The group G is the quotient group of $F(x_1, \cdots, x_k)$ by the normal subgroup generated by $R_1, \cdots, R_q$. Now we can construct a space $X = X(G)$ such that $\pi_1(X) = G$ as follows:

First we choose the wedges of circles, $X_1 = \vee^k S^1$ and $X_2 = \vee^q S^1$. Now we define a map $f\colon \vee^q S^1 \to \vee^k S^1$ such that f restricted to the j-th copy of S^1 is a representative of the element $R_j \in \pi_1(\vee^k S^1) = F(x_1, \cdots, x_k)$. Define

$$X = X_1 \coprod CX_2 / \sim,$$

where $\sim$ is the equivalence relation generated by

$$(0,x)\sim f(x)$$

for $x\in\vee^q S^1$. We show that $\pi_1(X)=G$. Let $x=1\times X_2$ be the element in $CX_2=I\times X_2/1\times X_2$, where $X_2=\vee^q S^1$. Let

$$U=X\setminus x=X_1\coprod(CX_2\setminus\{x\})/\sim.$$

Then $U\simeq X_1=\vee^k S^1$ and so $\pi_1(U)=F(x_1,\cdots,x_k)$. Let V be image of $(2/1,1]\times X_2$ in CX_2. Then V is an open neighborhood of x with $\pi_1(V)=0$ (V is contractible). Clearly that $U\cap V\simeq X_2=\vee^q S^1$. Thus $\pi_1(X)$ is the quotient group of $F(x_1,\cdots x_k)$ by the normal subgroup generated by

$$\mathrm{Im}(\pi_1(U\cap V)\to\pi_1(U))=\mathrm{Im}(f_*\colon\pi_1(\vee^q S^1)\to\pi_1(\vee^k S^1)),$$

which is the normal subgroup generated by $R_1,\cdots R_q$. Thus $\pi_1(X)=G$.

Now let $\phi\colon G\to H$ be a homomorphism. Suppose that G has generators $x_1,\cdots x_k$ with relations $R_1,\cdots R_q$ and H has generators $y_1,\cdots y_s$ with relations $S_1,\cdots S_t$. Then there is a homomorphism $\tilde\phi\colon F(x_1,\cdots,x_k)\to F(y_1,\cdots,y_t)$ such that the diagram

$$\begin{tikzcd}
G \arrow[r, "\phi"] & H \\
F(x_1,\cdots,x_k) \arrow[u, twoheadrightarrow] \arrow[r, "\tilde\phi"] & F(y_1,\cdots,y_s) \arrow[u, twoheadrightarrow]
\end{tikzcd}$$

commutes. Thus there is a map $f\colon X_1(G)\to X_1(H)$ such that

$$f_*=\tilde\phi\colon\pi_1(X_1(G))\to\pi_1(X_1(H)).$$

Let $j\colon X_1(H)\to X(H)$ be the inclusion. Then the composite

$$\theta\colon X_2(G)\longrightarrow X_1(G)\xrightarrow{f}X_1(H)\hookrightarrow X(H)$$

is null homotopic because its restriction to each copy of S^1 induces the trivial element in the fundamental group of $X(H)$. It follows that there is a map $\tilde\theta\colon CX_2(G)\to X(H)$ such that $\tilde\theta|_{X_2(G)}=\theta$. Now the map $j\circ f$ and $\tilde\theta$ defines a map

$$\bar f\colon X(G)=X_1(G)\coprod CX_2(G)/\sim\to X(H).$$

Clearly $\bar f_*=\phi\colon\pi_1(X(G))\to\pi_1(X(H))$. Thus we have the following theorem.

Theorem 3.5.7 *For any group G, there is a group $X(G)$ such that $\pi_1(X(G)) = G$. If $\phi: G \to H$ is a homomorphism, there is a map $f: X(G) \to X(H)$ such that*

$$f_* = \phi: \pi_1(X(G) = G \to \pi_1(X(H)) = H.$$

Note: The space $X(G)$ is not unique (even up to homotopy) because a group G can be written down in terms of different generator-relation systems.

Example 3.5.8 If a group G has only one relation (such group are called *one relator groups*, the construction of $X(G)$ is quite simple which can be described as follows:

Let $x_1, \cdots, x_k$ be generators for G and let $R = x_{i_1}^{\epsilon_1} \cdots x_{i_t}^{\epsilon_t}$ be the only relation for G. We may assume that R is an unreduced word such that all $x_1, \cdots, x_k$ occur in R.

Let Y be a t-sided polygonal region with counter-clockwise orientation. The sides in Y are labeled by $x_{i_1}, \cdots, x_{i_t}$. The j-th side is chosen to be in a positive direction [negative direction] if $\epsilon_j = 1$ [if $\epsilon_j = -1$].

Let X be the quotient space of Y by identifying 1) all vertices to be one point and 2) all oriented sides labeled by the same letter.

We can show that $\pi_1(X) = G$. Let x be an inner point in Y. Let $U = X \setminus \{x\}$. Then $U \simeq \vee^k S^1$. Let V be an open ϵ-neighborhood of x in Y (and so in X). Then $\pi_1(V) = 0$. Clearly $U \cap V \simeq S^1$ and so $\pi_1(U \cap V) = \mathbb{Z}$. Let $j: U \cap V \to U$ be the inclusion and let $\alpha = [\mathrm{id}_S^1]$ be the generator for $\pi_1(U \cap V)$. Then

$$j_*(\alpha) = x_{i_1}^{\epsilon_1} \cdots x_{i_t}^{\epsilon_t}.$$

Thus $\pi_1(X) = G$. ♠