INTRODUCTION TO ALGEBRAIC TOPOLOGY TUTORIAL 1

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Exercise 0.1. a) Show that each of the following is a metric for $\mathbb{R}^n$:

$$d(x,y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2} = ||x-y||; \quad d(x,y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y; \end{cases}$$

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|; \qquad d(x,y) = \max_{1\leq i\leq n} |x_i - y_i|.$$

b) Show that $d(x,y)$ does not define a metric on $\mathbb{R}$.
c) Show that $d(x,y) = \min_{1\leq i\leq n} |x_i - y_i|$ does not define a metric on $\mathbb{R}^n$.
d) Let d be a metric. Show that d' defined by

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

is also a metric.

Exercise 0.2. Show that if $\mathcal{U}$ is the family of open sets arising from a metric space then

i) The empty set $\emptyset$ and the whole set belong to $\mathcal{U}$;
ii) The intersection of two members of $\mathcal{U}$ belongs to $\mathcal{U}$;
iii) The union of any number of members of $\mathcal{U}$ belongs to $\mathcal{U}$.

Exercise 0.3. Let $\mathcal{U}$ be a topology for X. Show that the intersection of a finite number of members of $\mathcal{U}$ is in $\mathcal{U}$. Show by examples that infinite intersection of open sets in a topological space may not be open.

Exercise 0.4. *Let X be a metric space with metric d. Let d' be the new metric defined in Exercise 0.1. Then (X,d) and (X,d') has the same topology. (Hint: Show that the identity maps $\mathrm{id}_X\colon (X,d)\to(X,d')$ and $\mathrm{id}_X\colon (X,d')\to(X,d)$ are continuous by either using $\epsilon-\delta$-method or showing that the pre-image of open sets are open.)*

Exercise 0.5. Prove each of the following statements.

a) If Y is a subset of a topological space X with $Y \subseteq F \subseteq X$ and F is closed then $\bar{Y} \subseteq F$.

b) Y is closed if and only if $Y = \bar{Y}$.
c) $\bar{\bar{Y}} = \bar{Y}$.
d) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
e) $X \setminus \overset{\circ}{Y} = \overline{X \setminus Y}$.
f) $\bar{Y} = Y \cup \partial Y$ where $\partial Y = \bar{Y} \cap \overline{(X \setminus Y)}$ (∂Y is called the boundary of Y).
g) Y is closed if and only if $\partial Y \subseteq Y$.
h) $\partial Y = \emptyset$ if and only if Y is both open and closed.
i) For $a < b \in \mathbb{R}$

$$\partial(a, b) = \partial[a, b] = \{a, b\}.$$

Exercise 0.6. Show that

1) the subspace (a, b) of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. (Hint: Use functions like $x \to \tan(\pi(cx + d))$ for suitable c and d.)
2) the subspaces $(1, \infty), (0, 1)$ of $\mathbb{R}$ are homeomorphic. (Hint: $x \to 1/x$.)
3) $S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ is homeomorphic to $\mathbb{R}^n$ with the usual topology. (Hint: Define $\phi\colon S^n \setminus \{(0, 0, \cdots, 0, 1)\} \to \mathbb{R}^n$ by

$$\phi(x_1, x_2, \cdots, x_{n+1}) = (\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}} \cdots, \frac{x_n}{1 - x_{n+1}})$$

and $\psi\colon \mathbb{R}^n \to S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ by

$$\psi(x_1, \cdots, x_n) = \frac{1}{1 + \|x\|^2}(2x_1, 2x_2, \cdots, 2x_n, \|x\|^2 - 1).)$$

Harder Problems are given below.

Problem 1. For each $m, n \geq 0$, show that S^{m+n} is homeomorphic to $S^m \wedge S^n$.

Reference: C. R. F. Maunder, Algebraic Topology, Cambridge University Press, 1980, Page 208, Example 6.2.15.

Problem 2. Show that $S^n/(\mathbb{Z}/2) \cong \mathbb{R}P^n$ and $\mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n$.

Hint: You can do this problem by the following steps.

(1). Show that $\mathbb{R}P^n$ is Hausdorff. To prove this, let l_1 and l_2 be two elements in $\mathbb{R}P^n$, that is two lines in $\mathbb{R}^{n+1}$ passing the origin. Let $q\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the quotient map and let $x, y \in \mathbb{R}^{n+1}$ with $\|x\| = \|y\| = 1$, $x \in l_1$ and $y \in l_2$. Let ϵ be a positive number such that $\epsilon < \min\{\|x + y\|, \|x - y\|\}$. (Such an ϵ exists because x and y are linearly independent vectors.) Consider the open balls $B_{\epsilon/2}(x)$ and $B_{\epsilon/2}(y)$. Show that $q^{-1}(q(B_{\epsilon/2}(x)))$ and $q^{-1}(q(B_{\epsilon/2}(y)))$ are disjoint open sets in $\mathbb{R}^{n+1} \setminus \{0\}$. By this, you get that $q(B_{\epsilon/2}(x))$ and $q(B_{\epsilon/2}(y))$ are disjoint open neighborhoods of x and y, respectively. (So $\mathbb{R}P^n$ is Hausdorff by the definition.)

(2). Let $\pi\colon S^n \to \mathbb{R}P^n$ be the composite $S^n \hookrightarrow \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}P^n$, that is $\pi(x)$ is the line passing x and the origin. You find that $\pi(x) = \pi(-x)$. By using this, show that π induces a well-defined function $\bar\pi\colon S^n/(\mathbb{Z}/2) \to \mathbb{R}P^n$. Now you check that: 1) $\bar\pi$ is a map (by the definition of quotient topology), 2)$\bar\pi$ is onto and 3) $\bar\pi$ is one-to-one. Now you show that $\bar\pi$ is a homeomorphism as follows. Because S^n is compact, the quotient $S^n/(\mathbb{Z}/2)$ is compact. Show the general statement that any map from a compact space to a Hausdorff space is a closed map [This is assertion 5 of Exercise 2.8.4. Say $f\colon X \to Y$ is such a map. Let A be closed in the compact space X. Then A is compact (Theorem 2.8.3) and so $f(A)$ is compact in the Hausdorff space Y (Theorem 2.8.2). Thus $f(A)$ is closed (Theorem 2.8.5).] By this statement, $\bar\pi$ is a closed bijective continuous function and so $\bar\pi$ is a homeomorphism.

(3). Let $S^n_+ = \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \|x\| = 1 \text{ and } x_{n+1} \geq 0\}$. Let $\tilde\pi\colon S^n_+ \to \mathbb{R}P^n$ be the restriction of the map π, that is, $\tilde\pi(x)$ is the line passing x and the origin. Consider S^{n-1} as a subspace of S^n_+ by

$$S^{n-1} = \{x = (x_1, \cdots, x_n, 0) | \|x\| = 1\}.$$

Show that $\tilde\pi^{-1}(\mathbb{R}P^{n-1}) = S^{n-1}$ as a subspace of S^n_+. [Let $x \in S^n_+$. $\tilde\pi(x) \in \mathbb{R}P^{n-1}$ means that this line lies in the subspace $\mathbb{R}^n = \{(x_1, \cdots, x_n, 0)\} \subseteq \mathbb{R}^{n+1}$.] By pinching out S^{n-1} and $\mathbb{R}P^{n-1}$ from S^n_+ and $\mathbb{R}P^n$, respectively, the map $\tilde\pi$ induces a bijective (continuous) map $\pi'\colon S^n_+/S^{n-1} \to \mathbb{R}P^n/\mathbb{R}P^{n-1}$. Because S^n_+/S^{n-1} is compact and $\mathbb{R}P^n/\mathbb{R}P^{n-1}$ is Hausdorff [Corollary 2.8.8], π' is a homeomorphism. Now

$$S^n_+/S^{n-1} \cong D^n/S^{n-1} \cong S^n$$

and so this finishes the proof.