INTRODUCTION TO ALGEBRAIC TOPOLOGY TUTORIAL 6

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Exercise 0.1. Let T be a torus. Show that $\pi_1(T \setminus \{x\})$ is a free group of rank 2 for any $x \in T$.

Exercise 0.2. Let X be a space. The *unreduced suspension* $\Sigma_u X$ is the quotient space of $I \times X$ obtained by identifying $0 \times X$ to a point and $1 \times X$ to a (different) point. Suppose that X is path-connected. Show that $\Sigma_u X$ is simply connected.

Exercise 0.3. Show that $\mathbb{C}P^n$ is simply connected for each $n \geq 1$.

Exercise 0.4. Let $T_g = T\#T\#\cdots\#T$ be the g-fold connected sum of the torus T. Show that $\pi_1(T_g)$ is the quotient group of the free group $F(c_1, d_1, c_2, d_2, \cdots, c_g, d_g)$ by the one relation:

$$c_1 d_1 c_1^{-1} d_1^{-1} c_2 d_2 c_2^{-1} d_2^{-1} \cdots c_g d_g c_g^{-1} d_g^{-1} = 1.$$

Exercise 0.5. Let X be a quotient of the rectangle obtained by identifying opposite edges. Compute the fundamental group of X in all possible cases.

(Note: Let the edges of the rectangle be labeled by a_1, a_2, a_3 and a_4, where (a_1, a_3) and (a_2, a_4) are two pairs of oposite edges. There are two possibilities of the identification of a_1 and a_3, that is a_1 is identified with a_3 with the same direction OR with the opposite direction. Thus there are four possibilities for the identification and so four possible quotient spaces. Two of them are the torus, one is the projective plane and the other case is called the Klein bottle.)

Exercise 0.6. Compute the fundamental group of the space X.

i) $X = S^2 \coprod I/\sim$, where the equivalent relation $\sim$ is generated by $(1,0,0) \sim 0$ and $(-1,0,0) \sim 1$. That is X is the space obtained by adding extra line to the sphere from north to south.

ii)

$$X = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | \|x\| = 1\} \cup \{(t,0,0) \in \mathbb{R}^3 | 1 \leq t \leq 2\}$$
$$\cup \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | \|x\| = 2\} \cup \{(t,0,0) \in \mathbb{R}^3 | -2 \leq t \leq -1\}.$$