INTRODUCTION TO ALGEBRAIC TOPOLOGY
TUTORIAL 7

JIE WU

Exercise 0.1. Construct a space X (with proof) such that $\pi_1(X)$ is

i) the cyclic group $\mathbb{Z}/n$.
ii) generated by x, y with the relation $x^2 = y^3$.
iii) generated by x, y, z with the relations $x = [y, z], y = [x, z]$ and $z = [x, y]$, where $[a, b] = a^{-1}b^{-1}ab$ is called the *commutator* of elements a, b.

(Hint: Read Theorem 3.5.7 and Example 3.5.8.)

Exercise 0.2. Let $p\colon \tilde{X} \to X$ be a covering projection. Show that

i) p is an open map.
ii) If X is Hausdorff, then so is $\tilde{X}$.
iii) If X is an m-manifold, then so is $\tilde{X}$.
iv) If X is Hausdorff and $\tilde{X}$ is an m-manifold, then X is an m-manifold.
v) If $q\colon \tilde{Y} \to Y$ is also a covering projection, then so is $p \times q\colon \tilde{X} \times \tilde{Y} \to X \times Y$.

Exercise 0.3 (Pull-back)**.** Let $p\colon \tilde{X} \to X$ be a covering projection and let $f\colon Y \to X$ be a map. Let

$$\tilde{Y} = \{(y, \tilde{x}) \in Y \times \tilde{X} | f(y) = p(\tilde{x})\}$$

and let $q\colon \tilde{Y} \to Y$ be defined by $q(y, \tilde{x}) = y$. Show that q is a covering projection.

(Hint: Let $y \in Y$ and let U be an elementary neighbourhood of $f(y)$. Show that $f^{-1}(U)$ is an elementary neighbourhood of y.)

Exercise 0.4. Show that

i) Composition defines a group multiplication in the set $\mathrm{Homeo}(Y)$ of all homeomorphisms of a space Y to itself.
ii) Group actions $G \times Y \to Y$ correspond one-to-one with group homomorphisms $G \to \mathrm{Homeo}(Y)$.

The following two exercises are from the text book (pp. 148 Exercise 17.9 (e) and (f)).

Exercise 0.5. Let $a, b\colon \mathbb{C} \to \mathbb{C}$ be the homeomorphisms of the complex plane $\mathbb{C}$ defined by $a(z) = z + i$ and $b(z) = \bar{z} + 1/2 + i$. Show that $ba = a^{-1}b$ and deduce that

$$G = \{a^m b^{2n} b^{\epsilon} | m, n \in \mathbb{Z}\, \epsilon = 0 \text{ or } 1\}$$

is a group of homeomorphisms of $\mathbb{C}$. Furthermore, prove that the action of G is properly discontinuous and that the orbit space $\mathbb{C}/G$ is Hausdorff.

Exercise 0.6 (Continuation of Exercise 0.5)**.** Find a 'half-open rectangle' containing exactly one point from each orbit of G and hence show that $\mathbb{C}/G$ is a Klein bottle.