INTRODUCTION TO ALGEBRAIC TOPOLOGY TUTORIAL 8

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Exercise 0.1. Suppose that $\tilde{X}$ is path-connected. Show that the function $\psi\colon \pi_1(X, x_0) \to p^{-1}(x_0)$ is onto.

Hint: Let $y \in p^{-1}(x_0)$. There is a path β from $\tilde{x}_0$ to y. Let $\alpha = p \circ \beta$. Then $\beta = \tilde{\alpha}$ by the uniqueness of the lifting and so $\psi([\alpha]) = \tilde{\alpha}(1) = \beta(1) = y$.

Exercise 0.2. Let M be a path-connected manifold. Show that the configuration spaces $F(M, n)$ and $B(M, n)$ are path-connected.

Deduce that there is an epimorphism of groups $\phi\colon \psi_1(B(M, n)) \twoheadrightarrow \Sigma_n$ with $\mathrm{Ker}(\psi) \cong \pi_1(F(M, n))$, where Σ_n is the symmetric group.

Exercise 0.3. Prove that if $n \geq 2$ then there does not exists any continuous map $\phi\colon S^n \to S^1$ such that $\phi(-x) = -\phi(x)$.

Exercise 0.4. Let ζ be a primitive (complex) k-th root of the identity 1. Let $\mathbb{Z}/k$ acts on $\mathbb{C}^n$ by $\zeta \cdot (z_1, \cdots, z_n) = (\zeta z_1, \cdots, \zeta z_n)$ for $z_j \in \mathbb{C}$. Show that for $k, n \geq 2$ no $\mathbb{Z}/k$-map $f\colon \mathbb{C}^n \to \mathbb{C}$ can be norm-preserving, that is if $f(\zeta z) = \zeta f(z)$ for all $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, then for some $z \in \mathbb{C}^n$ $|f(z)| \neq \|z\|$.

Deduce that for $n \geq 2$ any map $h\colon \mathbb{C}^n \to \mathbb{C}$ has some $z \in \mathbb{C}^n$ such that $z \neq 0$ and

$$\sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z) = 0.$$

(Hint: i) Assume that there are such an $f\colon \mathbb{C}^n \to \mathbb{C}$. Then f induces a Z/k-map $\bar{f}\colon S^{2n-1} \to S^1$. Use the arguments in proof of Borsuk-Ulam Theorem to find a contradiction.

ii) Assume that $\sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z) \neq 0$ for all $z \neq 0$ in $\mathbb{C}^n$. Let

$$f(z) = \frac{\sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z)}{\| \sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z) \|}$$

for $z \in S^{2n-1}$. Check that f is a Z/k-map.)

Exercise 0.5. A map-maker wishes to represent the Earth's surface in an atlas. Explain how any representation of the Earth's surface on a sheet of paper which neither

i) excludes at least one point of the Earth's surface (e.g. North pole),
ii) includes some point of the Earth surface more than once, nor
iii) contains a 'cut', so that some nearby points on the Earth are noticeably separated on paper,

corresponds to an injective map from S^2 to $\mathbb{R}^2$. Hence show that no such representation is possible.

Exercise 0.6. Prove that

a) for $n \geq 2$ any continuous map $\mathbb{R}P^n \to S^1$ is null homotopic.
b) for $n \geq 1$ any continuous map $L^n(p) \to S^1$ is null homotopic, where $L^n(p)$ is the lens space.