MA2108 **Midterm** **18 March 2002**
Professor J. Wu

Time allowed: 1.5 hours

Tutorial Group:(circle one)

Tuesday 11-12	Tuesday 12-1	Tuesday 1-2
Wednesday 4-5	Friday 10-12	Friday 4-5

ID number:______________________________

Name:______________________________

Signature:______________________________

MA2108 Midterm **18 March** **Name:**____________

Problem #	Your Grades
1 (10 points)	
2 (10 points)	
3 (10 points)	
4 (10 points)	
5 (10 points)	
6 (10 points)	
7 (10 points)	
8 (10 points)	
9 (10 points)	
10 (10 points)	
total (100 points)	

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Determine the limit of the following sequences:

1. $\left\{\sin\left(\frac{3+n^2\pi+n}{5+n+2n^2}\right)+\ln\left(\frac{n^2+1}{4n+n^2+2}\right)\right\}$.

Solution.

$$\begin{aligned}&\lim_{n\to\infty}\sin\left(\frac{3+n^2\pi+n}{5+n+2n^2}\right)+\ln\left(\frac{n^2+1}{4n+n^2+2}\right)\\ =&\lim_{n\to\infty}\sin\left(\frac{3/n^2+\pi+1/n}{5/n^2+1/n+2}\right)+\ln\left(\frac{1+1/n^2}{4/n+1+2/n^2}\right)\\ =&\sin\frac{\pi}{2}+\ln 1=1+0=1.\end{aligned}$$

☐

2. $\left\{\frac{n^{50}50^n+n!+\ln n}{n!+2^n}\right\}$.

Solution.

$$\begin{aligned}&\lim_{n\to\infty}\frac{n^{50}50^n+n!+\ln n}{n!+2^n}\\ =&\lim_{n\to\infty}\frac{\frac{n^{50}}{50^n}\frac{100^n}{n!}+1+\frac{\ln n}{n!}}{1+\frac{2^n}{n!}}\\ =&\frac{0+1+0}{1+0}=1.\end{aligned}$$

☐

3. $\left\{\dfrac{\sqrt{2n+2}-\sqrt{n}}{\sqrt{n}}\right\}$.

Solution.

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\sqrt{2n+2}-\sqrt{n}}{\sqrt{n}}\\
&=\lim_{n\to\infty}\frac{(\sqrt{2n+2}-\sqrt{n})(\sqrt{2n+2}+\sqrt{n})}{\sqrt{n}(\sqrt{2n+2}+\sqrt{n})}\\
&=\lim_{n\to\infty}\frac{2n+2-n}{\sqrt{n}(\sqrt{2n+2}+\sqrt{n})}=\lim_{n\to\infty}\frac{1+2/n}{\sqrt{2+2/n}+1}\\
&=\frac{1}{\sqrt{2}+1}=\sqrt{2}-1.
\end{aligned}$$

□

4. $\left\{\left(1-\dfrac{2}{n+3}\right)^{2n}\right\}$.

Solution.

$$\begin{aligned}
&\lim_{n\to\infty}\left(1-\frac{2}{n+3}\right)^{2n}\\
&=\lim_{n\to\infty}\frac{\left(\left(1-\frac{2}{n+3}\right)^{n+3}\right)^2}{\left(1-\frac{2}{n+3}\right)^6}\\
&=\frac{\left(e^{-2}\right)^2}{(1-0)^6}=e^{-4}.
\end{aligned}$$

□

Determine convergence or divergence of the following series:

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+5n-2}{2n+n^4+1}$.

Proof. Let $a_n = \dfrac{3n^2+5n-2}{2n+n^4+1}$ and let $b_n = \dfrac{1}{n^2}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{3n^2+5n-2}{2n+n^4+1} \cdot n^2 \\ &= \lim_{n\to\infty} \frac{3+5/n-2}{2/n^3+1+1/n^4} = 3.\end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series, the series is convergent by the limit comparison test.

$\square$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{2n}\right)^{n^2}}{2^n}$.

Solution. Let $a_n = \dfrac{\left(1+\frac{1}{2n}\right)^{n^2}}{2^n}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{2n}\right)^{n}}{2} = \frac{\sqrt{e}}{2} < 1.$$

Thus the series is convergent by the root test. $\square$

Determine the absolute convergence, conditional convergence or divergence of the following series:

7. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{\sqrt{n}+1}$.

Solution. Let $a_n = \dfrac{\ln n}{\sqrt{n}+1}$. Then $a_n > 0$. Let $f(x) = \dfrac{\ln n}{\sqrt{n}+1}$. Then

$$f'(x) = \frac{\frac{1}{x}(\sqrt{x}+1) - \frac{\ln x}{2\sqrt{x}}}{(\sqrt{x}+1)^2} = -\frac{\ln x - 2 - \frac{2}{\sqrt{x}}}{2\sqrt{x}(\sqrt{x}+1)^2} \leq 0$$

when $\ln x > 4$ or $x > e^4$. Thus $\{a_n\}$ is (eventually) monotone decreasing. Moreover

$$\lim_{n\to\infty} a_n = \frac{\frac{\ln n}{\sqrt{n}}}{1+\frac{1}{\sqrt{n}}} = \frac{0}{1} = 0.$$

The series is convergent by the alternating series test. Now

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{\ln n}{\sqrt{n}+1}\right| = \sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n}+1}.$$

Let $b_n = \dfrac{1}{\sqrt{n}}$. Since $\displaystyle\lim_{n\to\infty}\frac{b_n}{a_n} = \lim_{n\to\infty}\frac{\sqrt{n}+1}{\sqrt{n}\ln n} = \lim_{n\to\infty}\frac{1+\frac{1}{\sqrt{n}}}{\ln n} = 0$, that is $b_n << a_n$, and $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent by the p-series, the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{\ln n}{\sqrt{n}+1}\right|$ is divergent by the limit comparison test for the case $a_n << b_n$ and hence the original series is conditionally convergent. $\square$

8. $\displaystyle\sum_{n=3}^{\infty}\frac{(-1)^{n+1}\cos n}{n\left((\ln n)^2-1\right)}$.

Solution. Observe that $\left|\dfrac{(-1)^{n+1}\cos n}{n\left((\ln n)^2+1\right)}\right| \leq \dfrac{1}{n\left((\ln n)^2-1\right)} := a_n$. Let $b_n = \dfrac{1}{n(\ln n)^2}$. Then $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n(\ln n)^2}{n\left((\ln n)^2-1\right)} = \lim_{n\to\infty}\frac{1}{1-\frac{1}{(\ln n)^2}} = 1$. By the example in the lecture notes, $\displaystyle\sum_{n=3}^{\infty} b_n$ is convergent and so is $\displaystyle\sum_{n=3}^{\infty} a_n$ by the limit comparison test. Now, by the comparison test, the series $\displaystyle\sum_{n=3}^{\infty}\left|\frac{(-1)^{n+1}\cos n}{n\left((\ln n)^2+1\right)}\right|$ is convergent and hence the original series is absolutely convergent. $\square$

9. Let $\{a_n\}$ be a bounded sequence of real numbers. Show that

$$\varlimsup_{n\to\infty} a_n^3 = \left(\varlimsup_{n\to\infty} a_n\right)^3.$$

Solution. Let $b_n = \sup\{a_n, a_{n+1}, \ldots\}$ and let $B_n = \sup\{a_n^3, a_{n+1}^3, \ldots\}$. First we show that $B_n = b_n^3$.

For each $k \geq n$, we have $a_k \leq b_n$ and so $a_k^3 \leq b_n^3$ for $k \geq n$. Thus b_n^3 is an upper bound of $\{a_n^3, a_{n+1}^3, \ldots\}$ and so $B_n \leq b_n^3$.

Conversely $a_k^3 \leq B_n$ for $k \geq n$ and so $a_k \leq \sqrt[3]{B_n}$ for $k \geq n$. Thus $\sqrt[3]{B_n}$ is an upper bound of $\{a_n, a_{n+1}, \ldots\}$ and so $b_n \leq \sqrt[3]{B_n}$ or $b_n^3 \leq B_n$.

Hence $B_n = b_n^3$ for all n and

$$\varlimsup_{n\to\infty} a_n^3 = \lim_{n\to\infty} B_n = \lim_{n\to\infty} b_n^3 = \left(\lim_{n\to\infty} b_n\right)^3 = \left(\varlimsup_{n\to\infty} a_n\right)^3.$$

$\square$

10. Let $\sum_{n=1}^{\infty} a_n$ be a series and let $\{b_n\}$ be a bounded sequence. Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Show that the series $\sum_{n=1}^{\infty} a_n b_n$ is also absolutely convergent.

Solution. Since $\{b_n\}$ is bounded, the exists M such that $|b_n| \leq M$ for all n. Thus

$$|a_n b_n| = |a_n||b_n| \leq |a_n|M$$

for all n. By the assumption, $\sum_{n=1}^{\infty} |a_n|$ is convergent. So is $\sum_{n=1}^{\infty} |a_n|M = M\sum_{n=1}^{\infty} |a_n|$. By the comparison test, the series $\sum_{n=1}^{\infty} |a_n b_n|$ is convergent and hence the result. ☐