

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2001-2002

MA2108 Advanced Calculus II

April 2002 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** the questions in this section. Section A carries a total of 60 marks.

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

- (a) $\left\{ \sqrt{\frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}} \right\}$.
- (b) $\left\{ (\sqrt{n} + 1)(\sqrt{3n + 1} - \sqrt{3n}) \right\}$.
- (c) $\left\{ \left(\frac{3n}{3n - 1} \right)^{2n} \right\}$.
- (d) $\left\{ \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \right\}$.

Solution. (a).

$$\lim_{n \rightarrow \infty} \sqrt{\frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}} = \lim_{n \rightarrow \infty} \sqrt{\frac{1 + 2n^5/n! + \ln n/n!}{1 + 5^n/n! + 3n/n!}} = 1.$$

(b).

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sqrt{n} + 1)(\sqrt{3n + 1} - \sqrt{3n}) &= \lim_{n \rightarrow \infty} \frac{\sqrt{n} + 1}{\sqrt{3n + 1} + \sqrt{3n}} \\ &= \lim_{n \rightarrow \infty} \frac{1 + 1/\sqrt{n}}{\sqrt{3 + 1/n} + \sqrt{3}} = \frac{1}{2\sqrt{3}}. \end{aligned}$$

(c).

$$\lim_{n \rightarrow \infty} \left(\frac{3n}{3n - 1} \right)^{2n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 - \frac{1}{3n}\right)^{2n}} = \frac{1}{\left(e^{-\frac{1}{3}}\right)^2} = e^{\frac{2}{3}}.$$

(d).

$$\left| \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \right| \leq \frac{n^{50} \cdot 50^n}{n!}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{n^{50} \cdot 50^n}{n!} = \lim_{n \rightarrow \infty} \frac{n^{50}}{2^n} \frac{100^n}{n!} = 0 \cdot 0 = 0,$$

$$\lim_{n \rightarrow \infty} \left| \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \right| = 0$$

by the Squeeze theorem and so

$$\lim_{n \rightarrow \infty} \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} = 0.$$

■

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{\ln n + 2n - 1}{3n^3 + 2n^2 + n + 1}.$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}.$

(c) $\sum_{n=1}^{\infty} 6^n \left(1 - \frac{2}{n+1}\right)^{n^2}.$

(d) $\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}.$

Solution. (a). Let $a_n = \frac{\ln n + 2n - 1}{3n^3 + 2n^2 + n + 1}$ and let $b_n = \frac{1}{n^2}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(\ln n + 2n - 1)n^2}{3n^3 + 2n^2 + n + 1} = \lim_{n \rightarrow \infty} \frac{\frac{\ln n}{n} + 2 - \frac{1}{n}}{3 + 2\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}} = \frac{2}{3}.$$

Since $\sum_{n=1}^{\infty} b_n$ converges by the p -series, the series $\sum_{n=1}^{\infty} a_n$ converges by the limit comparison test.

(b). Let $f(x) = \frac{1}{x(2 + \ln x)}$. Then $f(x)$ is positive, continuous and monotone decreasing on $[1, +\infty]$. Since

$$\begin{aligned} \int_1^{\infty} f(x) dx &= \int_1^{\infty} \frac{1}{x(2 + \ln x)} dx \\ &= \int_2^{\infty} \frac{dy}{y} = \ln y \Big|_2^{\infty} = +\infty \end{aligned}$$

diverges, the series $\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}$ diverges by the integral test.

(c). Let $a_n = 6^n \left(1 - \frac{2}{n+1}\right)^{n^2}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{a_n} &= \lim_{n \rightarrow \infty} 6 \left(1 - \frac{2}{n+1}\right)^n \\ &= \lim_{n \rightarrow \infty} 6 \frac{\left(1 - \frac{2}{n+1}\right)^{n+1}}{1 - \frac{2}{n+1}} = 6e^{-2} = \frac{6}{e^2} < 1 \end{aligned}$$

and so the series is convergent by the simplified root test.

(d). Let $a_n = \frac{n^n}{3^n \cdot n!}$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1} \cdot 3^n \cdot n!}{3^{n+1} \cdot (n+1)! \cdot n^n} \\ &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^n \cdot (n+1)}{3 \cdot (n+1)} = \frac{e}{3} < 1, \end{aligned}$$

the series is convergent by the ratio test.

■

Question 3 [12 marks]

(a) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{2n^2 + \ln n}.$$

(b) Does the series of functions

$$\sum_{n=1}^{\infty} n^2 x^n \sin nx$$

converge uniformly on the interval $[0, \frac{1}{2}]$? Justify your answer.

Solution. (a). The radius of convergence

$$R = \frac{1}{\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{2n^2 + \ln n}{2(n+1)^2 + \ln(n+1)}} = 1.$$

Consider the ending points: $x = x_0 \pm R = \pm 1$.

When $x = -1$, the series

$$\sum_{n=1}^{\infty} -\frac{1}{2n^2 + \ln n} = -\sum_{n=1}^{\infty} \frac{1}{2n^2 + \ln n}$$

is convergent by the limit comparison test because $\lim_{n \rightarrow \infty} \frac{\frac{1}{2n^2 + \ln n}}{\frac{1}{n^2}} = \frac{1}{2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

When $x = 1$, the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n^2 + \ln n}$$

is absolutely convergent because the series

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{2n^2 + \ln n} \right| = \sum_{n=1}^{\infty} \frac{1}{2n^2 + \ln n}$$

converges from the above.

Hence the interval of convergence is $[-1, +1]$.

(b).

$$\left| n^2 x^n \sin nx \right| \leq \frac{n^2}{2^n}$$

for $x \in [0, \frac{1}{2}]$. Since

$$\lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{2n^2} = \frac{1}{2} < 1,$$

the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges and so the series of functions $\sum_{n=1}^{\infty} n^2 x^n \sin nx$ converges uniformly on $[0, \frac{1}{2}]$ by the Weierstrass M -test. ■

Question 4 [16 marks]

(a) Find the general solutions of the following differential equations.

(i) $(1 + t^2)y' + ty = 0$.

(ii) $\frac{dy}{dt} = \frac{y}{t} + 2 \tan\left(\frac{y}{t}\right)$.

(b) Is the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n + 1}{\sqrt{n}}$ absolutely convergent, conditionally convergent or divergent? Justify your answer.

Solution. (a) (i).

$$\frac{dy}{dt} = -\frac{ty}{1+t^2} \Rightarrow \int \frac{dy}{y} = -\int \frac{t}{1+t^2} dt$$

$$\begin{aligned}\Rightarrow \ln|y| &= -\frac{1}{2}\ln(1+t^2) + k \\ \Rightarrow y &= \frac{C}{\sqrt{1+t^2}}.\end{aligned}$$

(ii). Let $v = \frac{y}{t}$. Then $y' = tv' + v$. From the equation,

$$\begin{aligned}tv' + v &= v + 2\tan v \quad \Rightarrow \quad \int \frac{\cos v}{\sin v} dv = \int \frac{2dt}{t} \\ \Rightarrow \ln|\sin v| &= 2\ln|t| + k \quad \Rightarrow \quad \sin v = Ct^2 \\ \Rightarrow \sin \frac{y}{t} &= Ct^2 \quad \Rightarrow \quad y = t \arcsin(Ct^2).\end{aligned}$$

(b). Conditional convergence. Let $a_n = \frac{\ln n + 1}{\sqrt{n}}$. Then $a_n \geq 0$ and

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} + \frac{1}{\sqrt{n}} = 0.$$

Let $f(x) = \frac{\ln x + 1}{\sqrt{x}}$ for $x \geq 1$. Then

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - (\ln x + 1)\frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x - 1}{2x^{\frac{3}{2}}} \leq 0$$

when $\ln x \geq 1$ or $x \geq e$. Thus $f(x)$ is eventually monotone decreasing and so is the sequence $\{a_n\}$. By the alternating series test, the series

$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n + 1}{\sqrt{n}}$ is convergent.

Consider the series

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln n + 1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} a_n.$$

Since $\frac{\ln n + 1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$ and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p -series,

the series $\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln n + 1}{\sqrt{n}} \right|$ diverges by the Comparison test.

Hence the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n + 1}{\sqrt{n}}$ is conditionally convergent. ■

SECTION B

Answer not more than **TWO (2)** questions from this section. Each question in this section carries 20 marks.

Question 5 [20 marks]

(a) Evaluate $\lim_{n \rightarrow \infty} \int_2^3 \frac{n^2 \cdot \ln x \cdot \sin nx}{x^n} dx$. Justify your answer.

(b) Consider the sequence $\{a_n\}$ defined recursively by

$$a_1 = \sqrt{3}, \quad a_{n+1} = \sqrt{3 + a_n}, \quad \text{for } n \geq 1.$$

Show that $\{a_n\}$ converges, and find its limit.

(c) Let $\{a_n\}$ be a bounded sequence of real numbers and let f be a continuous function on $(-\infty, +\infty)$. Show that

$$f\left(\overline{\lim}_{n \rightarrow \infty} a_n\right) \leq \overline{\lim}_{n \rightarrow \infty} f(a_n).$$

[Hint: Recall from tutorial that there is a subsequence of $\{a_n\}$ convergent to $\overline{\lim}_{n \rightarrow \infty} a_n$.]

Solution. (a). Since, for $2 \leq x \leq 3$,

$$\begin{aligned} \left| \frac{n^2 \cdot \ln x \cdot \sin nx}{x^n} \right| &\leq \frac{n^2 \cdot \ln x}{x^n} \\ \lim_{n \rightarrow \infty} \frac{n^2 \cdot \ln x}{x^n} &= 0, \\ \lim_{n \rightarrow \infty} \left| \frac{n^2 \cdot \ln x \cdot \sin nx}{x^n} \right| &= 0 \\ \Rightarrow F(x) = \lim_{n \rightarrow \infty} \frac{n^2 \cdot \ln x \cdot \sin nx}{x^n} &= 0. \end{aligned}$$

for $2 \leq x \leq 3$. Now

$$T_n = \sup_{2 \leq x \leq 3} |F_n(x) - F(x)| = \sup_{2 \leq x \leq 3} \left| \frac{n^2 \cdot \ln x \cdot \sin nx}{x^n} \right| \leq \sup_{2 \leq x \leq 3} \frac{n^2 \cdot \ln x}{x^n}.$$

Let $f(x) = \frac{n^2 \cdot \ln x}{x^n}$. Then

$$f'(x) = n^2 \cdot \left(\frac{1}{x} x^{-n} - n \ln x x^{-n-1} \right) = \frac{n^2(1 - n \ln x)}{x^{n+1}}.$$

Thus $f'(x)$ exists and is nowhere zero on $[2, 3]$ and so

$$0 \leq T_n \leq \sup_{2 \leq x \leq 3} \frac{n^2 \cdot \ln x}{x^n} = \max_{2 \leq x \leq 3} \frac{n^2 \cdot \ln x}{x^n} = \max\{f(2), f(3)\} = \frac{n^2 \ln 2}{2^n}.$$

Since $\lim_{n \rightarrow \infty} \frac{n^2 \ln 2}{2^n} = 0$, $\lim_{n \rightarrow \infty} T_n = 0$ by the Squeeze theorem and so $\left\{ \frac{n^2 \cdot \ln x \cdot \sin nx}{x^n} \right\}$ converges uniformly on $[2, 3]$. Thus

$$\lim_{n \rightarrow \infty} \int_2^3 \frac{n^2 \cdot \ln x \cdot \sin nx}{x^n} dx = \int_2^3 \lim_{n \rightarrow \infty} \frac{n^2 \cdot \ln x \cdot \sin nx}{x^n} dx = \int_2^3 0 dx = 0.$$

(b).

$$\begin{aligned} \sqrt{101} &= \sqrt{100 + 1} = 10 \left(1 + \frac{1}{100} \right)^{\frac{1}{2}} \\ &= 10 \left(1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{100^n} \right) \\ &= 10 \left(1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2} \cdot (\frac{1}{2} - 1) \cdots (\frac{1}{2} - n + 1)}{n! \cdot 10^{2n}} \right) \\ &= 10 \left(1 + \frac{1}{200} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2n-3)!!}{n! \cdot 2^n \cdot 10^{2n}} \right) \end{aligned}$$

Let $a_n = \frac{(2n-3)!!}{n! \cdot 2^n \cdot 10^{2n}}$. Then $a_n > 0$. We check that $\{a_n\}$ is monotone decreasing and $\lim_{n \rightarrow \infty} a_n = 0$. Since

$$\frac{a_{n+1}}{a_n} = \frac{(2n-1)!! \cdot n! \cdot 10^{2n}}{(2n-3)!! \cdot (n+1)! \cdot 2^{n+1} \cdot 10^{2n+2}} = \frac{(2n-1)}{(n+1) \cdot 2 \cdot 100} < 1,$$

$\{a_n\}$ is monotone decreasing. Now from

$$0 \leq a_n \leq \frac{1}{10^{2n}}$$

and $\lim_{n \rightarrow \infty} \frac{1}{10^{2n}} = 0$, $\lim_{n \rightarrow \infty} a_n = 0$ by the Squeeze theorem. Note that $a_2 = \frac{1}{8} \times 10^{-4}$ and $a_3 = \frac{1}{16} \times 10^{-6}$ with

$$10 \cdot a_3 = \frac{1}{16} \times 10^{-5} < 10^{-6}.$$

By the Alternating test estimation, we have

$$\sqrt{101} \approx 10 \left(1 + \frac{1}{200} - \frac{1}{8} \times 10^{-4} \right) = 10 + \frac{1}{20} - \frac{1}{8000} = 10.049875$$

with error within 10^{-6} .

(c). By Question 8 in Tutorial 3, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k \rightarrow \infty} a_{n_k} = \overline{\lim}_{n \rightarrow \infty} a_n$. Thus

$$f(\overline{\lim}_{n \rightarrow \infty} a_n) = f(\lim_{k \rightarrow \infty} a_{n_k}) = \lim_{k \rightarrow \infty} f(a_{n_k})$$

because f is continuous. Since $\{f(a_{n_k})\}$ is a subsequence of $\{f(a_n)\}$, we have

$$\lim_{k \rightarrow \infty} f(a_{n_k}) \leq \overline{\lim}_{n \rightarrow \infty} f(a_n)$$

by Part (b) of Question 8 in Tutorial 3. Hence

$$f(\overline{\lim}_{n \rightarrow \infty} a_n) = f(\lim_{k \rightarrow \infty} a_{n_k}) = \lim_{k \rightarrow \infty} f(a_{n_k}) \leq \overline{\lim}_{n \rightarrow \infty} f(a_n).$$

■

Question 6 [20 marks]

(a) Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{x e^{-nx}}{n}.$$

Is $f(x)$ continuous on $(0, +\infty)$? Justify your answer.

(b) Consider the sequence $\{a_n\}$ defined recursively by

$$a_1 = \sqrt{3}, \quad a_{n+1} = \sqrt{3 + a_n}, \quad \text{for } n \geq 1.$$

Show that $\{a_n\}$ converges, and find its limit.

(c) Let $\{a_n\}$ be a **convergent** sequence of real numbers. Suppose that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is **conditionally** convergent. Show that

$$(i) \quad \lim_{n \rightarrow \infty} a_n = 0.$$

$$(ii) \quad \overline{\lim}_{n \rightarrow \infty} n^p |a_n| = +\infty \text{ for any } p > 0.$$

Solution. (a). Let $f_n(x) = \frac{x e^{-nx}}{n^2}$.

$$f'_n(x) = \frac{e^{-nx} - n x e^{-nx}}{n} = 0 \quad \Rightarrow \quad x = \frac{1}{n}$$

$$f_n(0) = \lim_{x \rightarrow \infty} f_n(x) = 0.$$

Thus

$$f_n(x) \leq \sup_{0 < x < +\infty} f_n(x) = \max_{0 < x < +\infty} f_n(x) = f_n\left(\frac{1}{n}\right) = \frac{e^{-1}}{n^2}$$

for $0 < x < +\infty$. Since $\sum_{n=1}^{\infty} \frac{e^{-1}}{n^2} = e^{-1} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p -series,

$\sum_{n=1}^{\infty} f_n(x)$ converges uniformly and so $f(x)$ is continuous.

(b). First we show that $0 \leq a_n \leq 3$ by induction. When $n = 1$, $0 \leq a_1 = \sqrt{3} \leq 3$. Suppose that $0 \leq a_n \leq 3$. Then

$$0 \leq \sqrt{3} \leq \sqrt{a_n + 3} = a_{n+1} \leq \sqrt{6} \leq 3.$$

Hence $0 \leq a_n \leq 3$ for all n and $\{a_n\}$ is bounded.

Next we show that a_n is monotone increasing. Clearly $a_1 \leq a_2$. Suppose that $a_{n-1} \leq a_n$ with $n \geq 2$. Then

$$a_{n+1} - a_n = \sqrt{a_n + 3} - \sqrt{a_{n-1} + 3} = \frac{a_n - a_{n-1}}{\sqrt{a_n + 3} + \sqrt{a_{n-1} + 3}} \geq 0.$$

Thus $\{a_n\}$ is monotone increasing and $\lim_{n \rightarrow \infty} a_n$ exists. Let $A = \lim_{n \rightarrow \infty} a_n$. Then

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{a_n + 3} = \sqrt{\lim_{n \rightarrow \infty} a_n + 3} = \sqrt{A + 3} \\ &\Rightarrow A^2 = A + 3 \quad \Rightarrow A = \frac{1 \pm \sqrt{13}}{2}. \end{aligned}$$

Since $a_n \geq 0$ for all n , $A \geq 0$. Hence

$$A = \frac{\sqrt{13} + 1}{2}.$$

(c) (i). Let $A = \lim_{n \rightarrow \infty} a_n$. Suppose that $A \neq 0$.

Case I. $A > 0$. Then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is eventually positive. Since

$$\lim_{n \rightarrow \infty} \frac{\frac{a_n}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} a_n \neq 0, +\infty$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p -series, the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ diverges by the limit comparison test. This contradicts to that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Case II $A < 0$. Let $b_n = -a_n$. Then $\lim_{n \rightarrow \infty} b_n = -A > 0$. By Case I, $\sum_{n=1}^{\infty} \frac{b_n}{n}$ diverges and so does $\sum_{n=1}^{\infty} \frac{a_n}{n} = -\sum_{n=1}^{\infty} \frac{b_n}{n}$. This contradicts to that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Hence one gets a contradiction in either Case I or Case II and so $\lim_{n \rightarrow \infty} a_n = A = 0$.

(c) (ii). Suppose that there exists $p > 0$ such that $\overline{\lim}_{n \rightarrow \infty} n^p |a_n| < +\infty$. Then $\{n^p |a_n|\}$ is a bounded sequence and so there exists $M > 0$ such that $n^p |a_n| \leq M$ for all n . It follows that

$$\left| \frac{a_n}{n} \right| \leq \frac{M}{n^{1+p}}.$$

Since the series $\sum_{n=1}^{\infty} \frac{M}{n^{1+p}} = M \sum_{n=1}^{\infty} \frac{1}{n^{1+p}}$ is convergent by the p -series, the series

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n} \right|$$

converges and so the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges absolutely. This contradict

to that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is conditionally convergent and hence the result. ■

Question 7 [20 marks]

(a) Evaluate the limit

$$\lim_{t \rightarrow 0} \frac{\sin(t^2) - \arctan(t^2)}{1 - \cos(t^3)}.$$

(b) Let $\sum_{n=1}^{\infty} f_n(x)$ be the series of **positive** functions on an interval I and let $\{G_n(x)\}$ be a sequence of functions on I such that

$$\sup_{x \in I} |G_n(x)| < +\infty$$

for each $n \geq 1$. Suppose that the series of functions $\sum_{n=1}^{\infty} f_n(x)$ and the sequence of functions $\{G_n(x)\}$ converge uniformly to some function $G(x)$ on I .

(i) Show that

$$\sup_{x \in I} |G(x)| < +\infty$$

(ii) Use (i) or otherwise, show that there exists a positive number M such that

$$|G_n(x)| \leq M$$

for all $x \in I$ and all $n \geq 1$.

(ii) Using (ii) or otherwise, show that the series of functions

$$\sum_{n=1}^{\infty} f_n(x)G_n(x) \text{ converges uniformly on } I.$$

Solution. (a). From the Taylor series

$$\begin{aligned} \sin t &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n-1}}{(2n-1)!} \\ \arctan t &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n-1}}{2n-1} \\ \cos t &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}, \\ \lim_{t \rightarrow 0} \frac{\sin(t^2) - \arctan(t^2)}{1 - \cos(t^3)} \\ &= \lim_{t \rightarrow 0} \frac{\left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \dots\right) - \left(t^2 - \frac{t^6}{3} + \frac{t^{10}}{5} - \dots\right)}{1 - \left(1 - \frac{t^6}{2!} + \frac{t^{12}}{4!}\right)} \\ &= \lim_{t \rightarrow 0} \frac{\left(-\frac{1}{3!} + \frac{1}{5}\right)t^6 + \left(\frac{1}{5!} - \frac{1}{5}\right)t^{10} + \dots}{\frac{1}{2}t^6 - \frac{1}{4!}t^{12} + \dots} = \frac{1}{3}. \end{aligned}$$

(b) (i). Since $\{G_n(x)\}$ converges uniformly to $G(x)$ on I , there exists N such that

$$|G_n(x) - G(x)| < 1$$

$\Rightarrow |G(x)| = |G(x) - G_n(x) + G_n(x)| \leq |G(x) - G_n(x)| + |G_n(x)| < |G_n(x)| + 1$
for $n > N$ and $x \in I$. Choose $n = N + 1$.

$$\sup_{x \in I} |G(x)| \leq \sup_{x \in I} |G_{N+1}(x)| + 1 < +\infty.$$

(ii). There exists N such that

$$\left| G_n(x) - G(x) \right| < 1$$

$\Rightarrow |G_n(x)| = |G(x) - G_n(x) + G_n(x)| \leq |G(x) - G_n(x)| + |G_n(x)| < |G(x)| + 1$
for $n > N$ and $x \in I$. Let

$$M = \max\left\{ \sup_{x \in I} |G_j(x)|, 1 + \sup_{x \in I} |G(x)| \mid 1 \leq j \leq N_1 \right\}.$$

Then $|G_n(x)| \leq M$ for $n \leq N$ and

$$\left| G_n(x) \right| \leq 1 + \left| G(x) \right| \leq M$$

for $n > N$.

(b) (iii). By the assumption, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I . Let M be given in (ii). Given any $\epsilon > 0$, there exists N such that

$$\left| \sum_{k=1}^n f_n(x) - \sum_{k=1}^{n+p} f_n(x) \right| = \sum_{k=n+1}^{n+p} f_k(x) < \frac{\epsilon}{M}$$

for $x \in I$, $n > N$ and $p \geq 1$. When $n > N$ and $p \geq 1$, we have

$$\begin{aligned} & \left| \sum_{k=1}^n f_k(x)G_k(x) - \sum_{k=1}^{n+p} f_k(x)G_k(x) \right| = \left| \sum_{k=n+1}^{n+p} f_k(x)G_k(x) \right| \\ & \leq \sum_{k=n+1}^{n+p} |f_k(x)G_k(x)| = \sum_{k=n+1}^{n+p} f_k(x)|G_k(x)| \leq M \sum_{k=n+1}^{n+p} f_k(x) < M \cdot \frac{\epsilon}{M} = \epsilon \end{aligned}$$

and hence the series $\sum_{n=1}^{\infty} f_k(x)G_k(x)$ converges uniformly on I because the sequence of its partial sums converges uniformly by the Cauchy Criteria. ■