NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2001-2002

MA2108 Advanced Calculus II

November 2001 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

(1) 1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

(2) 2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.

(3) 3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

(4) 4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A
Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\sqrt[3]{\dfrac{8^n + n^8 + n!}{n! + (\ln n)^8}}\right\}$.

(b) $\left\{\dfrac{\sqrt{n}}{\sqrt{2n+1} - \sqrt{n}}\right\}$.

(c) $\left\{\left(\dfrac{2n}{2n-1}\right)^{3n}\right\}$.

(d) $\left\{3n \sin\left(\dfrac{1}{n}\right) + (5^n + 3^n)^{\frac{1}{n}}\right\}$.

Solution. (a).

$$\lim_{n\to\infty} \sqrt[3]{\frac{8^n + n^8 + n!}{n! + (\ln n)^8}} = \lim_{n\to\infty} \sqrt[3]{\frac{\frac{8^n}{n!} + \frac{n^8}{n!} + 1}{1 + \frac{(\ln n)^8}{n!}}} = \sqrt[3]{\frac{0+0+1}{1+0}} = 1.$$

(b). $$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1} - \sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}(\sqrt{2n+1} + \sqrt{n})}{(\sqrt{2n+1} - \sqrt{n})(\sqrt{2n+1} + \sqrt{n})}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2n^2+n} + n}{2n+1-n} = \lim_{n\to\infty} \frac{\sqrt{2 + \frac{1}{n}} + 1}{1 + \frac{1}{n}} = \frac{\sqrt{2+0}+1}{1+0} = \sqrt{2} + 1.$$

(c).

$$\lim_{n\to\infty} \left(\frac{2n}{2n-1}\right)^{3n} = \lim_{n\to\infty} \left[\frac{1}{\left(1 + \frac{-\frac{1}{2}}{n}\right)^n}\right]^3 = \left(e^{\frac{1}{2}}\right)^3 = e^{\frac{3}{2}}.$$

(d).

$$\lim_{n\to\infty} 3n \sin\left(\frac{1}{n}\right) + (5^n + 3^n)^{\frac{1}{n}} = \lim_{n\to\infty} 3\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} + 5\left[1 + \left(\frac{3}{5}\right)^n\right]^{\frac{1}{n}}$$
$$= 3 \cdot 1 + 5 \cdot (1+0)^0 = 8.$$

∎

Question 2 [16 marks]

Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{2\sqrt{n}+1}$.

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1+\sin n}{2n^2-n}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{4n}(n!)^3}{(3n)!}$.

Solution. (a). Divergence. Since

$$\lim_{n\to\infty} \frac{\frac{1}{n+\ln n}}{\frac{1}{n}} = 1$$

and the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the series is divergent by the limit comparison test.

(b). Conditional convergence. Let $a_n = \dfrac{1}{2\sqrt{n}+1}$. Then the positive sequence $\{a_n\}$ is monotone decreasing and $\displaystyle\lim_{n\to\infty} a_n = 0$. By the alternating series test, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{2\sqrt{n}+1}$ is convergent. Since $a_n \geq \dfrac{1}{3\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by the p-series, the series $\displaystyle\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{2\sqrt{n}+1}\right|$ is divergent by the comparison test and so the series in the statement is conditionally convergent.

(c). Absolute convergence. Observe $\left|(-1)^n \dfrac{1+\sin n}{2n^2-n}\right| \leq \dfrac{2}{2n^2-n} \leq \dfrac{2}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series, the series

$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1+\sin n}{2n^2-n}\right|$ is convergent by the comparison test and the series is absolutely convergent.

(d) (Absolute) convergence because

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{4(n+1)}[(n+1)!]^3(3n)!}{[3(n+1)]!2^{4n}(n!)^3}$$

$$=\lim_{n\to\infty}\frac{2^4(n+1)^3}{(3n+3)(3n+2)(3n+1)}=\frac{16}{27}<1$$

and by the ratio test. ∎

Question 3 [12 marks]

Find the interval of convergence of the following power series.

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2-\ln n}$.

(b) $\sum_{n=1}^{\infty}\frac{(2x-1)^n}{\sqrt{n}}$.

Solution. (a). Let $a_n=(-1)^{n+1}\frac{1}{n^2-\ln n}$. From

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{(n^2-\ln n)}{(n+1)^2-\ln(n+1)}=1,$$

the radius of convergence $R=1$. When $x=\pm1$ the ending points, since the series $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{(\pm1)^n}{n^2-\ln n}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2-\ln n}$ is convergent by the limit comparison test with the p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2-\ln n}$ converges absolutely at the end points $x=\pm1$ and so the interval of convergence is $[-1,1]$.

(b). $\sum_{n=1}^{\infty}\frac{(2x-1)^n}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{2^n}{\sqrt{n}}\left(x-\frac{1}{2}\right)^n$. Let $a_n=\frac{2^n}{\sqrt{n}}$. From

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{2^{n+1}\sqrt{n}}{\sqrt{n+1}2^n}=2,$$

the radius of converges $R = \frac{1}{2}$. We check the ending points $x = x_0 \pm R = 0, 1$. When $x = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent by the alternating series test. When $x = 1$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by the p-series. Thus the interval of convergence is $[0, 1)$. ∎

Question 4 [16 marks]

(a) Find the general solutions of the following differential equations.

(i) $y' - x^2 y = y$.

(ii) $y'' + 5y' + 6y = 0$.

(b) Evaluate the limit: $\lim_{t \to 0} \frac{1 - \cos(t^5)}{t \sin(t^3) - t^4}$.

Solution. (a) (i). From $\frac{dy}{dx} = (x^2 + 1)y$, we have $\frac{1}{y} dy = (x^2 + 1) dx$ and so $\int \frac{1}{y} dy = \int (x^2 + 1) dx$. It follows that

$$\ln |y| = \frac{1}{3} x^3 + x + C \quad \text{or} \quad y = C e^{\frac{1}{3} x^3 + x}.$$

(a) (ii). From $r^2 + 5r + 6 = 0$, we have $r = -2$ or -3. It follows that $y(t) = C_1 e^{-2t} + C_2 e^{-3t}$.

(b). By the Taylor series, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and so

$$\sin t^3 = t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!} - \cdots \qquad \cos t^5 = 1 - \frac{t^{10}}{2!} + \frac{t^{20}}{4!} - \cdots .$$

Thus

$$\lim_{t \to 0} \frac{1 - \cos(t^5)}{t \sin(t^3) - t^4} = \lim_{t \to 0} \frac{\frac{t^{10}}{2!} - \frac{t^{20}}{4!} + \cdots}{t(t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!} - \cdots) - t^4}$$

$$= \lim_{t \to 0} \frac{\frac{1}{2!} - \frac{t^{10}}{4!} + \cdots}{-\frac{1}{3!} + \frac{t^6}{5!} - \cdots} = \frac{\frac{1}{2!}}{-\frac{1}{3!}} = -3.$$

∎

SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Evaluate $\displaystyle\lim_{n\to\infty}\int_0^1\left(\frac{x^2+x+1}{4}\right)^n \sin nx dx$. Justify your answer.

(b) Find the first three terms (up to and including the quadratic term) for the Taylor series for the function $f(x) = \sqrt[3]{x}$ expanded around $x = 8$.

(c) Use an appropriate series to find a decimal approximation to $\sqrt[3]{8.1}$ that is accurate to within 10^{-4}. Justify your answer.

Solution. (a). Let $F_n(x) = \left(\frac{x^2+x+1}{4}\right)^n \sin nx$. For $0 \le x \le 1$, we have

$$|F_n(x)| = \left|\frac{x^2+x+1}{4}\right|^n |\sin nx| \le \left(\frac{3}{4}\right)^n.$$

Since $\displaystyle\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$, $F(x) = \displaystyle\lim_{n\to\infty} F_n(x) = 0$ for $0 \le x \le 1$ by the Squeeze Theorem. Observe that

$$T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1}\left|\frac{x^2+x+1}{4}\right|^n |\sin nx| \le \left(\frac{3}{4}\right)^n.$$

Since $\displaystyle\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$, we have $\displaystyle\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem. Thus the sequence $\{F_n(x)\}$ converges uniformly to $F(x)$ and so

$$\lim_{n\to\infty}\int_0^1\left(\frac{x^2+x+1}{4}\right)^n \sin nx dx = \int_0^1 \lim_{n\to\infty}\left(\frac{x^2+x+1}{4}\right)^n \sin nx dx = 0.$$

(b). Observe that $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ and $f''(x) = -\frac{1}{3}\frac{2}{3}x^{-\frac{5}{3}}$. We obtain $f(8) = 8^{\frac{1}{3}} = 2$, $f'(8) = \frac{1}{3}8^{-\frac{2}{3}} = \frac{1}{12}$ and $f''(8) = -\frac{2}{9}8^{-\frac{5}{3}} = -\frac{2}{9\cdot 32} = -\frac{1}{144}$. Thus

$$\sqrt[3]{x} = 2 + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 + \cdots = 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288} + \cdots.$$

(c). Observe that

$$\sqrt[3]{8.1} = \left(8 + \frac{1}{10}\right)^{\frac{1}{3}} = 2\left(1 + \frac{1}{80}\right)^{\frac{1}{3}}.$$

By applying the binomial series $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, we have

$$\sqrt[3]{8.1} = 2\left(1 + \frac{1}{80}\right)^{\frac{1}{3}} = 2\sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} \frac{1}{80^n}.$$

Observe that

$$\binom{\frac{1}{3}}{n}\frac{1}{80^n} = \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\cdots\left(\frac{1}{3}-n+1\right)}{n!\cdot 80^n}$$

$$= (-1)^{n-1}\frac{2\cdot 5\cdot 8\cdots(3n-4)}{n!\cdot 3^n\cdot 80^n}$$

Let $a_0 = 1$, $a_1 = \frac{1}{240}$ and $a_n = \frac{2\cdot 5\cdot 8\cdots(3n-4)}{n!\cdot 3^n\cdot 80^n}$ for $n \geq 2$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2\cdot 5\cdot 8\cdots(3n-4)(3n-1)\cdot n!\cdot 80^n}{(n+1)!\cdot 3^{n+1}\cdot 80^{n+1}\cdot 2\cdot 5\cdot 8\cdots(3n-4)} = \frac{3n-1}{(n+1)240} < 1$$

and so $\{a_n\}$ is monotone decreasing. Now $a_2 = \frac{2}{2\cdot 240^2} = \frac{1}{240^2}$ and $a_3 = \frac{2\cdot 5}{6\cdot 240^3} = \frac{1}{6\cdot 2.4^3}10^{-5} < \frac{1}{2}\cdot 10^{-6}$. By the alternating series test estimation, we have

$$\sqrt[3]{8.1} \approx 2\left(1 + \frac{1}{240} - \frac{1}{240^2}\right) = 2 + \frac{1}{120} - \frac{1}{240\cdot 120},$$

where the error is within 10^{-6}. ∎

Question 6 [20 marks]

(a) Solve the following initial value problem:

$$y'' + 2y' + y = te^{-2t}; \quad y(0) = 0 \quad y'(0) = 2.$$

(b) Consider the sequence $\{a_n\}$ given by

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}, \quad \text{for n} \geq 1.$$

Show that $\{a_n\}$ converges, and find its limit.

(c) Does the series of functions $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^x}$ converge uniformly on the interval $(0,+\infty)$? Justify your answer.

Solution. (a). From $r^2+2r+1=0$, we have $r_1=r_2=-1$ and so the general solution for the homogenous equation $y''+2y'+y=0$ is

$$y_h(t)=C_1te^{-t}+C_2e^{-t}.$$

Let $y_p(t)=(A_0+A_1t)e^{-2t}$. Then

$$y_p'(t)=A_1e^{-2t}-2(A_0+A_1t)e^{-2t}=(A_1-2A_0)e^{-2t}-2A_1te^{-2t},$$

$$y_p''(t)=-2A_1e^{-2t}+4(A_0+A_1t)e^{-2t}-2A_1e^{-2t}=(4A_0-4A_1)e^{-2t}+4A_1te^{-2t}.$$

From $y''+2y'+y=te^{-2t}$, we have

$$(4A_0-4A_1)e^{-2t}+4A_1te^{-2t}+(2A_1-4A_0)e^{-2t}-4A_1te^{-2t}+A_0e^{-2t}+A_1te^{-2t}=te^{-2t}$$

and so

$$\begin{cases} 4A_0-4A_1+2A_1-4A_0+A_0=0 \\ 4A_1-4A_1+A_1=1. \end{cases}$$

Thus $A_1=1$ and $A_0=2A_1=2$. It follows that $y_p=2e^{-2t}+te^{-2t}$ and so the general solution is

$$y=C_1te^{-t}+C_2e^{-t}+2e^{-2t}+te^{-2t}.$$

Now $y'(t)=C_1e^{-t}-C_1te^{-t}-C_2e^{-t}-4e^{-2t}+e^{-2t}-2te^{-2t}$. We have

$$\begin{cases} 0+C_2+2+0=0 \\ C_1-0-C_2-4+1-0=2 \end{cases}$$

and so $C_2=-2$ and $C_1=3$. Answer: $y=3te^{-t}-2e^{-t}+2e^{-2t}+te^{-2t}$.

(b). First we show that $\sqrt{2}\le a_n\le 2$ by induction on n. This statement hold for $n=1$. Suppose that $\sqrt{2}\le a_{n-1}\le 2$ with $n\ge 2$. Then

$$\sqrt{2}\le\sqrt{2+a_{n-1}}=a_n\le\sqrt{2+2}=2.$$

The induction is finished and hence $\sqrt{2}\le a_n\le 2$.

Now we show that $\{a_n\}$ is monotone increasing by induction on n. Clearly $a_1\le a_2$. Suppose that $a_{n-1}\le a_n$. Then

$$a_{n+1}-a_n=\sqrt{2+a_n}-\sqrt{2+a_{n-1}}=\frac{a_n-a_{n-1}}{\sqrt{2+a_n}+\sqrt{2+a_{n-1}}}\ge 0.$$

The induction is finished and so $\{a_n\}$ is bounded and monotone decreasing. Thus the limit of $\{a_n\}$ exists. Let $A = \lim_{n\to\infty} a_n$. Then we have

$$A = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2 + \lim_{n\to\infty} a_n} = \sqrt{2+A}$$

and so $A^2 = 2 + A$ or $A = -1$ or 2. Since $a_n \geq \sqrt{2}$ for each n, we have $A \geq \sqrt{2}$ and so $A = 2$.

(c). Answer: NO. Let $S_n(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k^x}$. Suppose that the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x}$ converges uniformly on $(0, +\infty)$. Then the sequence $\{S_n(x)\}$ converges uniformly on $(0, +\infty)$ and so there exists N such that

$$|S_n(x) - S_m(x)| < \frac{1}{2}$$

for all $n, m > N$ and $x \in (0, +\infty)$. In particular,

$$\frac{1}{(N+2)^x} = |S_{N+2}(x) - S_{N+1}(x)| < \frac{1}{2}$$

for all $x \in (0, +\infty)$. This contradicts to that

$$\frac{1}{(N+2)^x} \geq \frac{1}{2}$$

when $0 < x \leq \frac{\ln 2}{\ln(N+2)}$. ∎

Question 7 [20 marks]

(a) Let $\{a_n\}$ be a bounded sequence of real numbers. Show that $\limsup_{n\to\infty} \sqrt{|a_n|} = \sqrt{\limsup_{n\to\infty} |a_n|}$.

(b) Let $a > 1$ be any positive number greater than 1. Show that the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^x}$ converges uniformly over the interval $[a, +\infty)$.

(c) Use (b) or otherwise, show that the function $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ is differentiable on $(1, +\infty)$.

Solution. (a). Let $b_n = \sup_{k\geq n} |a_k|$ and $B_n = \sup_{k\geq n} \sqrt{|a_k|}$. Since $\sqrt{|a_k|} \leq B_n$ for $k \geq n$, we have $|a_k| \leq B_n^2$ for $k \geq n$ and so $b_n \leq B_n^2$ or $\sqrt{b_n} \leq B_n$. It follows that

$$\sqrt{\limsup_{n\to\infty} |a_n|} = \sqrt{\lim_{n\to\infty} b_n} = \lim_{n\to\infty} \sqrt{b_n} \leq \lim_{n\to\infty} B_n = \limsup_{n\to\infty} \sqrt{|a_n|}.$$

Conversely, since $|a_k| \leq b_n$ for $k \geq n$, we have $\sqrt{|a_k|} \leq \sqrt{b_n}$ for $k \geq n$ and so $B_n \leq \sqrt{b_n}$. It follows that

$$\limsup_{n\to\infty} \sqrt{|a_n|} = \lim_{n\to\infty} B_n \leq \lim_{n\to\infty} \sqrt{b_n} = \sqrt{\lim_{n\to\infty} b_n} = \sqrt{\limsup_{n\to\infty} |a_n|}$$

and hence the result.

(b). Choose p such that $a > p > 1$. Let $M_n = \frac{\ln n}{n^a}$. Then the series $\sum_{n=1}^{\infty} M_n$ is convergent because

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^a}}{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{\ln n}{n^{a-p}} = 0,$$

that is, $\frac{\ln n}{n^a} << \frac{1}{n^p}$ by the Standard limits and the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Since $\left|\frac{\ln n}{n^x}\right| \leq M_n$ for $x \geq a$, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^x}$ converges uniformly on $[a, +\infty)$ by the M-test.

(c). Let x_0 be any point in $(1, +\infty)$. There exist numbers a and b such that $1 < a < x_0 < b$. Since

(1) the derivative $\left(\frac{1}{n^x}\right)' = \frac{-\ln n}{n^x}$ exists and is continuous on $[a, b]$.

(2) the series $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ converges on $[a, b]$.

(3) by (b) the series $\sum_{n=1}^{\infty} \frac{-\ln n}{n^x}$ converges uniformly on $[a, b]$,

the function $\zeta(x)$ is differentiable on $[a, b]$. In particular, $\zeta(x)$ is differentiable at x_0. Because x_0 is any given point in $(1, +\infty)$, the function $\zeta(x)$ is differentiable on $(1, +\infty)$.

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