1 i). $f'(x) = 2e^{2x}$, $f''(x) = 2^2e^{2x}$ and in general $f^{(k)}(x) = 2^ke^{2x}$. Thus

$$e^{2x} \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(3)}{k!}(x-3)^k = \sum_{k=0}^{\infty} \frac{2^ke^6}{k!}(x-3)^k = e^6\sum_{k=0}^{\infty} \frac{2^k}{k!}(x-3)^k.$$

2 ii). Since $f(x) = \cos x$, we have

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = f(x) = \cos x.$$

In general,

$$f^{(k)}(x) = \begin{cases} \cos x & \text{if} \quad k = 4l \\ -\sin x & \text{if} \quad k = 4l+1 \\ -\cos x & \text{if} \quad k = 4l+2 \\ \sin x & \text{if} \quad k = 4l+3 \end{cases}$$

and so

$$f^{(k)}\left(\frac{\pi}{3}\right) = \begin{cases} \frac{1}{2} & \text{if} \quad k = 4l \\ -\frac{\sqrt{3}}{2} & \text{if} \quad k = 4l+1 \\ -\frac{1}{2} & \text{if} \quad k = 4l+2 \\ \frac{\sqrt{3}}{2} & \text{if} \quad k = 4l+3 \end{cases}$$

Thus

$$\cos x \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(\frac{\pi}{3})}{k!}\left(x - \frac{\pi}{3}\right)^k,$$

where $f^{(k)}(\frac{\pi}{3})$ is given above.

2. From

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n},$$

$$\ln(1+2x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(2x^2\right)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot 2^n}{n}x^{2n}$$

for $|x| < \frac{1}{2}$.

3. Let $f(x) = \cos x$. Then

$$f^{(k)}(x) = \begin{cases} \cos x & \text{if} \quad k = 4l \\ -\sin x & \text{if} \quad k = 4l+1 \\ -\cos x & \text{if} \quad k = 4l+2 \\ \sin x & \text{if} \quad k = 4l+3 \end{cases}$$

By the Taylor Formula, we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}.$$

Observe that

$$0 \leq |R_n(x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!}|x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}.$$

By the Standard Limits, $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$. Thus $\lim\limits_{n\to\infty} |R_n(x)| = 0$ by the Squeeze Theorem and so $\lim\limits_{n\to\infty} R_n(x) = 0$. Hence

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \lim_{n\to\infty} f(x) - R_n(x) = f(x) - 0 = f(x).$$

It follows that

$$\cos x = f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

4.

$$\int_0^{0.2} \sin x^2 dx = \int_0^{\frac{1}{5}} \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x^2)^{2n-1}}{(2n-1)!}dx = \sum_{n=1}^{\infty} \int_0^{\frac{1}{5}} (-1)^{n+1}\frac{x^{4n-2}}{(2n-1)!}dx$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left(\frac{1}{5}\right)^{4n-1}}{(2n-1)! \cdot 4n-1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)! \cdot (4n-1) \cdot 5^{4n-1}},$$

where $\int_0^{\frac{1}{5}} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \int_0^{\frac{1}{5}}$ because the series of functions $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n-2}}{(2n-1)!}$ converges uniformly on $\left[0, \frac{1}{5}\right]$ by Theorem 7.5. Let $a_n = \frac{1}{(2n-1)! \cdot (4n-1) \cdot 5^{4n-1}}$. Then a_n is positive, monotone decreasing and $\lim\limits_{n\to\infty} a_n = 0$. By applying the alternating series estimation, from

$$a_{n+1} = \frac{1}{(2n+1)! \cdot (4n+3) \cdot 5^{4n+3}} < 10^{-8},$$

we have $n \geq 2$. Thus

$$\int_0^{0.2} \sin x^2 dx \approx \frac{1}{3 \cdot 5^3} - \frac{1}{3! \cdot 7 \cdot 5^7} = \frac{1}{375} - \frac{1}{3281250} \approx 0.002666362$$

with error $< 10^{-8}$.

5. i) Recall that

$$\arctan y = y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots,$$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots,$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots .$$

$$\lim_{y\to\infty} \frac{\arctan y - \sin y}{y^3 \cos y} = \lim_{y\to\infty} \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}{y^3\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)}$$

$$= \lim_{y\to\infty} \frac{\left(-\frac{1}{3} + \frac{1}{3!}\right) + \left(\frac{1}{5} - \frac{1}{5!}\right)y^2 + \cdots}{1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots} = -\frac{1}{3} + \frac{1}{3!} = -\frac{1}{6}.$$

5. ii) Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots .$$

$$\lim_{x\to\infty} x^2(e^{-1/x^2} - 1) = \lim_{x\to\infty} x^2\left(1 - \frac{1}{x^2} + \frac{1}{2!x^4} - \cdots - 1\right) = \lim_{x\to\infty} -1 + \frac{1}{2!x^2} - \cdots = -1.$$

6. Let $S_n(x) = \sum_{k=1}^{n} |f_k(x)|$ and let $T_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$. Since $\sum_{k=1}^{\infty} |f_k(x)|$ converges uniformly, the sequence of functions $\{S_n(x)\}$ converges uniformly. For any $\epsilon > 0$, there exists a natural number N such that

$$|S_n(x) - S_m(x)| < \frac{\epsilon}{M} \quad \text{for all} \quad x \in I \quad \text{and} n > m > N$$

$$\Rightarrow ||f_{m+1}(x)| + |f_{m+2}(x)| + \cdots + |f_n(x)|| < \frac{\epsilon}{M} \quad \text{for all} \quad x \in I \quad \text{and} n > m > N$$

$$\Rightarrow |f_{m+1}(x)| + |f_{m+2}(x)| + \cdots + |f_n(x)| < \frac{\epsilon}{M} \quad \text{for all} \quad x \in I \quad \text{and} n > m > N.$$

For $x \in I$ and $n > m > N$, we have

$$\begin{aligned}
|T_n(x) - T_m(x)| &= |f_{m+1}(x)g_{m+1}(x) + f_{m+2}(x)g_{m+2}(x) + \cdots + f_n(x)g_n(x)| \\
&\leq |f_{m+1}(x)g_{m+1}(x)| + |f_{m+2}(x)g_{m+2}(x)| + \cdots + |f_n(x)g_n(x)| \\
&\leq |f_{m+1}(x)|M + |f_{m+2}(x)|M + \cdots + |f_n(x)|M \\
&= M(|f_{m+1}(x)| + |f_{m+2}(x)| + \cdots + |f_n(x)|) < M \cdot \frac{\epsilon}{M} = \epsilon.
\end{aligned}$$

Thus the sequence of functions $\{T_n(x)\}$ converges uniformly and so the series of functions $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly.