2 a).

$$y = Ce^{\int -\sin t dt} = Ce^{\cos t}$$

with $C \in \mathbb{R}$. □

2 b). From $(1+t^2)\dfrac{dy}{dt} + ty = (1+t^2)^{3/2}$, we have $\dfrac{dy}{dt} + \dfrac{t}{1+t^2}y = (1+t^2)^{1/2}$. The integrating factor

$$\mu(t) = e^{\int g(t)dt} = e^{\int \frac{t}{1+t^2}dt} = e^{\frac{1}{2}\ln(1+t^2)} = (1+t^2)^{1/2}.$$

Thus the general solution

$$\begin{aligned} y &= \frac{1}{\mu(t)}\left(\int \mu(t)f(t)dt + C\right) = \frac{1}{(1+t^2)^{1/2}}\left(\int (1+t^2)^{1/2}\cdot(1+t^2)^{1/2}dt + C\right) \\ &= \frac{1}{(1+t^2)^{1/2}}\left(\int 1+t^2 dt + C\right) = \frac{1}{(1+t^2)^{1/2}}\left(t + \frac{1}{3}t^3 + C\right). \end{aligned}$$

□

3 a). The integrating factor

$$\mu(t) = e^{\int g(t)dt} = e^{\int 4tdt} = e^{2t^2}.$$

Thus the general solution

$$\begin{aligned} y &= \frac{1}{\mu(t)}\left(\int \mu(t)f(t)dt + C\right) = \frac{1}{e^{2t^2}}\left(\int 2te^{2t^2}\cdot e^{-2t^2}dt + C\right) \\ &= \frac{1}{e^{2t^2}}\left(\int 2tdt + C\right) = e^{-2t^2}(t^2 + C). \end{aligned}$$

From $y(1) = 5$, we have

$$5 = e^{-2}(1+C) \quad \text{or} \quad C = 5e^2 - 1$$

and so $y(t) = e^{-2t^2}(t^2 + 5e^2 - 1)$. □

3 b). From $(1+t^2)\dfrac{dy}{dt} - 2ty^4 = 0$, we have

$$\begin{aligned} \int \frac{dy}{y^4} &= \int \frac{2t}{1+t^2}dt + C \\ -\frac{y^{-3}}{3} &= \ln(1+t^2) + C. \end{aligned}$$

Since $y(0) = \dfrac{1}{2}$, $-\dfrac{\left(\frac{1}{2}\right)^{-3}}{3} = \ln(1+0) + C$ or

$$C = -\frac{8}{3}.$$

Thus $\dfrac{y^{-3}}{-3} = \ln(1+t^2) - \dfrac{8}{3}$ or

$$y^3 = \frac{1}{8 - 3\ln(1+t^2)}$$

and so $y = \dfrac{1}{\sqrt[3]{8 - 3\ln(1+t^2)}}$ □

3 c) From $\sqrt{1+t^3}\dfrac{dy}{dt} - 6t^2e^{-y} = 0$, we have

$$\int e^y dy = \int \frac{6t^2}{\sqrt{1+t^3}}dt + C = 2\int \frac{d(1+t^3)}{\sqrt{1+t^3}} + C = 2 \cdot 2\sqrt{1+t^3} + C.$$

Thus

$$e^y = 4\sqrt{1+t^3} + C.$$

Since $y(0) = 0$, $e^0 = 4\sqrt{1+0} + C$, and so $C = -3$. Hence $e^y = 4\sqrt{1+t^3} - 3$ or $y = \ln(4\sqrt{1+t^3} - 3)$. □

4 a). Let $v = \dfrac{y}{t}$. Then $y = vt$ and so $y' = v't + v$. From the equation $y' = \dfrac{y}{t} + \dfrac{y^2}{t^2}$, we have

$$tv' + v = v + v^2 \quad \text{or} \quad tv' = v^2.$$

Thus

$$\int \frac{dv}{v^2} = \int \frac{dt}{t} + C$$

$$-\frac{1}{v} = \ln(|t|) + C \quad \text{or} \quad -\frac{t}{y} = \ln(|t|) + C$$

and so $y = -\dfrac{t}{\ln(|t|) + C}$ □

4 b) Let $v = \dfrac{y}{t}$. Then $y = vt$ and so $y' = v't + v$. From the equation $y' = \dfrac{y}{t} - \tan\dfrac{y}{t}$, we have

$$tv' + v = v - \tan v.$$

Thus

$$\int \frac{cosvdv}{\sin v} = -\int \frac{dt}{t} + k$$

$$\int \frac{d\sin v}{\sin v} = -\ln|t| + k$$

$$\ln|\sin v| = -\ln|t| + k$$

$$|\sin v| = e^{-\ln|t| + k} = e^k \cdot \frac{1}{|t|}$$

$$\sin v = \frac{C}{t}$$

and so $\sin\dfrac{y}{t} = \dfrac{C}{t}$ or $y = t\left(2k\pi + \arcsin\left(\dfrac{C}{t}\right)\right)$ □

5 a). From
$$r^2 - 7r + 10 = 0,$$
we have $r = 2, 5$ and so $y = Ae^{2t} + Be^{5t}$ □

5 b). From
$$r^2 + 6r + 9 = 0,$$
we have $(r+3)^2 = 0$ or $r_1 = r_2 = -3$ and so $y = Ae^{-3t} + Bte^{-3t}$ □

5 c). From
$$r^2 + 2r + 3 = 0,$$
we have
$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 3}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2}i.$$
Thus $y = Ae^{-t}\cos\sqrt{2}t + Be^{-t}\sin\sqrt{2}t.$ □

6. From
$$r^2 + 2r + 1 = 0,$$
we have $(r+1)^2 = 0$ or $r = r_1 = r_2 = -1$. Thus $y = Ae^{-t} + Bte^{-t}$ and
$$y' = -Ae^{-t} - Bte^{-t} + Be^{-t}.$$
Since $y(0) = 1$ and $y'(0) = 0$, we have
$$\begin{cases} Ae^0 + B \cdot 0 \cdot e^0 = 1 \\ -Ae^0 - B \cdot 0 \cdot e^0 + Be^0 = 0. \end{cases}$$
and so $A = 1$ and $B = A = 1$. The solution is $y = e^{-t} + te^{-t}$. □

7 a). Since $c = 3 \neq 0$, try
$$y_p = At^3 + Bt^2 + Ct + D.$$
Then $y_p' = 3At^2 + 2Bt + C$ and $y_p'' = 6At + 2B$. From the equation $y_p'' + 3y_p = 6t^3 - 6$, we have
$$6At + 2B + 3(At^3 + Bt^2 + Ct + D) = 6t^3 - 6$$
and so
$$\begin{cases} 3A = 6 \\ 3B = 0 \\ 6A + 3C = 0 \\ 2B + 3D = -6. \end{cases}$$
Thus $A = 2$, $B = 0$, $C = -4$ and $D = -2$, and so $y_p = 2t^3 - 4t - 2$ □

7 b). From $r^2 + 3 = 0$, we have $r = \pm\sqrt{-3} = \pm\sqrt{3}i$. Try
$$y_p = t \cdot [A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)].$$
Then
$$y_p' = A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t) + t[-A\sin(\sqrt{3}t)\sqrt{3} + B\cos(\sqrt{3}t)\sqrt{3}]$$
$$\begin{aligned} y_p'' &= 2[-A\sin(\sqrt{3}t)\sqrt{3} + B\cos(\sqrt{3}t)\sqrt{3}] + t[-A\cos(\sqrt{3}t)(\sqrt{3})^2 - B\sin(\sqrt{3}t)(\sqrt{3})^2] \\ &= 2[-A\sin(\sqrt{3}t)\sqrt{3} + B\cos(\sqrt{3}t)\sqrt{3}] - 3t[A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)] \end{aligned}$$

Since $y_p'' + 3y_p = \cos(\sqrt{3}t)$, we have

$$2[-A\sin(\sqrt{3}t)\sqrt{3} + B\cos(\sqrt{3}t)\sqrt{3}] - 3t[A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)]$$
$$+3t \cdot [A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)] = \cos(\sqrt{3}t)$$

and so

$$\begin{cases} -2A\sqrt{3} = 0 \\ 2B\sqrt{3} = 1 \end{cases}$$

It follows that $y_p = \dfrac{1}{2\sqrt{3}} t\sin(\sqrt{3}t)$ □

8 a). From $r^2 - 3r + 2 = 0$, we have $(r-2)(r-1) = 0$ and so $r = 1, 2$. It follows that

$$y_h = Ae^t + Be^{2t}.$$

Try

$$y_p = t \cdot A_0 e^{2t} + B_0 e^{3t}.$$

We have

$$y_p' = 2A_0te^{2t} + A_0e^{2t} + 3B_0e^{3t}$$
$$y_p'' = 2A_0e^{2t} + 4A_0te^{2t} + 2A_0e^{2t} + 9B_0e^{3t}$$
$$= 4A_0te^{2t} + 4A_0e^{2t} + 9B_0e^{3t}.$$

Since $y_p'' - 3y_p' + 2y_p = 3e^{2t} + e^{3t}$, we have

$$4A_0te^{2t} + 4A_0e^{2t} + 9B_0e^{3t} - 3\left(2A_0te^{2t} + A_0e^{2t} + 3B_0e^{3t}\right) + 2\left(A_0te^{2t} + B_0e^{3t}\right) = 3e^{2t} + e^{3t}$$
$$(4A_0 - 3A_0)\,e^{2t} + (9B_0 - 9B_0 + 2B_0)\,e^{3t} = 3e^{2t} + e^{3t}$$

Thus $A_0 = 3$ and $B_0 = \dfrac{1}{2}$, and so $y(t) = y_h + y_p = Ae^t + Be^{2t} + 3te^{2t} + \dfrac{1}{2}e^{3t}$ □

8 b). From $r^2 + 4 = 0$, $r = \pm\sqrt{-4} = \pm 2i$. Thus $y_h = A\cos 2t + B\sin 2t$. Try

$$y_p = t(A_0\cos 2t + B_0\sin 2t) + C_0\sin t + D_0\cos t.$$

We have

$$y_p' = A_0\cos 2t + B_0\sin 2t + t(-2A_0\sin 2t + 2B_0\cos 2t) + C_0\cos t - D_0\sin t$$
$$y_p'' = -2A_0\sin 2t + 2B_0\cos 2t + (-2A_0\sin 2t + 2B_0\cos 2t)$$
$$+t(-4A_0\cos 2t - 4B_0\sin 2t) - C_0\sin t - D_0\cos t$$
$$= -4A_0\sin 2t + 4B_0\cos 2t - 4t(A_0\cos 2t + B_0\sin 2t) - C_0\sin t - D_0\cos t.$$

Since $y_p'' + 4y_p = \cos 2t + \sin t$,

$$-4A_0\sin 2t + 4B_0\cos 2t - 4t(A_0\cos 2t + B_0\sin 2t) - C_0\sin t - D_0\cos t$$
$$+4[t(A_0\cos 2t + B_0\sin 2t) + C_0\sin t + D_0\cos t] = \cos 2t + \sin t$$
$$-4A_0\sin 2t + 4B_0\cos 2t + 3C_0\sin t + 3D_0\cos t = \cos 2t + \sin t.$$

Thus $A_0 = 0$, $B_0 = \dfrac{1}{4}$, $C_0 = \dfrac{1}{3}$, and $D_0 = 0$, and so

$y = y_h + y_p = A\cos 2t + B\sin 2t + \dfrac{1}{4}t\sin 2t + \dfrac{1}{3}\sin t$ □

8 c). From $r^2 - 2r + 1 = 0$, we have $(r+1)^2 = 0$ or $r = r_1 = r_2 = 1$. Thus $y_1 = e^t$, $y_2 = te^t$ and

$$y_h = Ae^t + Bte^t.$$

Note that $a = 1$ and $g(t) = \dfrac{e^t}{t}$.

$$\begin{aligned} y_p(t) &= \left(\int \frac{gy_2}{ay_1'y_2 - ay_1y_2'}dt\right) \cdot y_1 + \left(\int \frac{-y_1g}{ay_1'y_2 - ay_1y_2'}dt\right) \cdot y_2 \\ &= \left(\int \frac{\frac{e^t}{t}te^t}{e^tte^t - e^t(te^t + e^t)}dt\right) \cdot e^t + \left(\int \frac{-e^t \cdot \frac{e^t}{t}}{e^tte^t - e^t(te^t + e^t)}dt\right) \cdot te^t \\ &= \int (-1)dt \cdot e^t + \int \frac{1}{t}dt \cdot te^t = -te^t + te^t \ln|t|. \end{aligned}$$

Thus

$$y(t) = Ae^t + Bte^t - te^t + te^t \ln|t| = Ae^t + Cte^t + te^t \ln|t|,$$

where $C = B - 1$ is any constant and so is A. $\square$

9. From $r^2 + 4r + 3 = 0$, we have $(r+3)(r+1) = 0$ or $r = -1, -3$. Thus

$$y_h = Ae^{-t} + Be^{-3t}.$$

Try

$$y_p = t \cdot A_0e^{-3t}.$$

We have

$$\begin{aligned} y_p' &= A_0e^{-3t} + A_0t)e^{-3t} \cdot (-3) = (A_0 - 3A_0t)e^{-3t} \\ y_p'' &= -3A_0e^{-3t} + (A_0 - 3A_0t)e^{-3t} \cdot (-3) = (-6A_0 + 9A_0t)e^{-3t}. \end{aligned}$$

Since $y_p'' + 4y_p' + 3y_p = 2e^{-3t}$,

$$\begin{aligned} (-6A_0 + 9A_0t)e^{-3t} + 4(A_0 - 3A_0t)e^{-3t} + 3A_0te^{-3t} &= 2e^{-3t} \\ -2A_0e^{-3t} &= 2e^{-3t}. \end{aligned}$$

Thus $A_0 = -1$ and $y(t) = y_h + y_p = Ae^{-t} + Be^{-3t} - te^{-3t}$, and

$$y' = -Ae^{-t} - 3Be^{-3t} - e^{-3t} + 3te^{-3t}.$$

Since $y(0) = 0$ and $y'(0) = 3$,

$$\begin{cases} Ae^0 + Be^0 - 0 = 0 \\ -Ae^0 - 3Be^0 - e^0 + 0 = 3. \end{cases}$$

Thus $B = -2$ and $A = 2$, and so $y(t) = 2e^{-t} - 3e^{-3t} - te^{-3t}$ $\square$

10. First we set up a differential equation. Let $y(t)$ denote the concentration of salt in the tank at time t. At time t we have

$$200 \cdot y(t) \quad \text{lb}$$

of salt. From time t to $t + \Delta t$, the salt in the tank roughly increases

$$\frac{1}{2} \quad \text{lb} \quad \times 4 \quad \text{gal/min} \quad \times \Delta t \quad \text{min}$$

and roughly decreases

$$y(t) \quad \text{lb/gal} \quad \times 4 \quad \text{gal/min} \quad \times \Delta t \quad \text{min.}$$

Thus we have

$$200y(t+\Delta t) - 200y(t) \approx \frac{1}{2} \cdot 4 \cdot \Delta t - y(t) \cdot 4 \cdot \Delta t$$

$$200\frac{y(t+\Delta t) - y(t)}{\Delta t} \approx \frac{1}{2} \cdot 4 - y(t) \cdot 4.$$

Let Δt tend to 0. We obtain the differential equation

$$200\frac{dy}{dt} = 2 - 4y \quad \text{or} \quad \frac{dy}{dt} + \frac{1}{50}y = \frac{1}{100}.$$

Now we solve this differential equation. From $\frac{dy}{dt} + \frac{1}{50}y = 0$, we have

$$y_h = Ce^{-\int \frac{1}{50}dt} = Ce^{-\frac{1}{50}t}.$$

Observe that $y_p = \frac{1}{2}$ is a solution of the nonhomogeneous equation. Thus the general solution is

$$y(t) = y_h + y_p = Ce^{-\frac{1}{50}t} + \frac{1}{2}.$$

Since $y(0) = \frac{S_0}{200}$,

$$\frac{S_0}{200} = C \cdot e^0 + \frac{1}{2}.$$

Thus $C = \frac{S_0}{200} - \frac{1}{2}$ and so $y(t) = \frac{S_0}{200}e^{-\frac{1}{50}t} + \frac{1}{2}\left(1 - e^{-\frac{1}{50}t}\right)$ □