2001/2002 Semester II MA2108 Advanced Calculus II

Solutions to Tutorial 2

Question 1. Answer: $\sup S = 1$ and $\inf S = 0$. We show that $\sup S = 1$. Let r be any element in S. By the definition, r is rational number with $0 \le r < 1$. Thus 1 is an upper bound of S and 0 is a lower bound of S. Let M be any upper bound of S. Then $r \le M$ for any rational number r with $0 \le r < 1$. In particular, $\frac{n}{n+1} \le M$ for any positive integer n. It follows that $1 = \lim_{n\to\infty} \frac{n}{n+1} \le M$ and so 1 is the least upper bound of S or $\sup S = 1$. Let m be any lower bound of S. Then $m \le r$ for any rational number r with $0 \le r < 1$. In particular, $m \le 0$ and so 0 is the greatest lower bound of S or $\inf S = 0$. □

Question 2. Since $\inf B$ is a lower bound of B, we have $\inf B \le x$ for any $x \in B$ and so $\inf B \le y$ for any $y \in A \subseteq B$. It follows that $\inf B$ is a lower bound of A. Thus $\inf B \le \inf A$ because $\inf A$ is the greatest lower bound of A. □

Question 3 (i). First we show that $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$. Let z be any element in $A \cup B$. Then $z \in A$ or B. If $z \in A$, then $z \le \sup A \le \max\{\sup A \sup B\}$. Otherwise, $z \in B$ and $z \le \sup B \le \max\{\sup A, \sup B\}$. Thus $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$.

Now we show that $\max\{\sup A, \sup B\}$ is the least upper bound of $A \cup B$. Let M be any upper bound of $A \cup B$. Then $z \le M$ for any $z \in A \cup B$. In particular, $z \le M$ for $z \in A \subseteq A \cup B$ and so M is an upper bound of A. It follows that $\sup A \le M$. Similarly, we have $\sup B \le M$. Thus $\max\{\sup A, \sup B\} \le M$ and so $\max\{\sup A, \sup B\} = \sup A \cup B$. □

Question 3 (ii). No, it is not true. An counter-example is as follows. Let $A = \{1, 2\}$ and let $B = \{1, 3\}$. Then $\sup A = \max A = 2$ and $\sup B = \max B = 3$. It follows that $\min\{\sup A, \sup B\} = \min\{2, 3\} = 2$. But $\sup A \cap B = \sup\{1\} = 1 \ne \min\{\sup A, \sup B\}$. □

Question 5. First we show that $0 \le x_n \le 1$ by induction on n. When $n = 1$, we have $0 \le x_1 = \frac{3}{4} \le 1$. Suppose that $0 \le x_{n-1} \le 1$ with $n \ge 2$. Observe that

$$x_n = 2x_{n-1} - x_{n-1}^2 = 1 - (1 - 2x_{n-1} + x_{n-1}^2) = 1 - (1 - x_{n-1})^2.$$

Since $0 \le 1 - x_{n-1} \le 1$ by induction, we have $0 \le x_n \le 1$. The induction is finished and so $0 \le x_n \le 1$ for all n.

Observe that

$$x_{n+1} - x_n = (2x_n - x_n^2) - x_n = x_n - x_n^2 = x_n(1 - x_n).$$

Since $0 \le x_n \le 1$, we have $x_{n+1} - x_n = x_n(1 - x_n) \ge 0$ and so the sequence $\{x_n\}$ is monotone increasing and bounded. Thus the limit of $\{x_n\}$ exists. Let $A = \lim_{n\to\infty} x_n$.

From the equation $x_{n+1} = 2x_n - x_n^2$, we have

$$A = \lim_{n\to\infty} x_{n+1} = 2\lim_{n\to\infty} x_n - \left(\lim_{n\to\infty} x_n\right)^2 = 2A - A^2.$$

and so $A = 0$ or 1. Since $x_n \geq x_1 = \dfrac{3}{4}$, we have $A = \lim_{n\to\infty} x_n \geq \dfrac{3}{4}$ and so $A \neq 0$. Thus $A = 1$. $\square$

Question 6 (a). Since $\{a_n\} = \{4 + \cos\frac{n\pi}{2}\} = \{4, 3, 4, 5, 4, 3, 4, 5, 4, 3, 4, 5, \cdots\}$, we have

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} = 5 \quad \text{and} \quad c_n = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\} = 3$$

for all n and so $\overline{\lim_{n\to\infty}}\, 4 + \cos\dfrac{n\pi}{2} = \lim_{n\to\infty} b_n = 5$ and $\underline{\lim}_{n\to\infty}\, 4 + \cos\dfrac{n\pi}{2} = \lim_{n\to\infty} c_n = 3$. $\square$

Question 6 (b). Observe that

$$0 \leq \frac{1 + (-1)^n}{n} \leq \frac{2}{n}.$$

Since $\lim_{n\to\infty} \dfrac{2}{n} = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} \dfrac{1 + (-1)^n}{n} = 0$ and so

$$\overline{\lim_{n\to\infty}} \frac{1 + (-1)^n}{n} = \underline{\lim}_{n\to\infty} \frac{1 + (-1)^n}{n} = \lim_{n\to\infty} \frac{1 + (-1)^n}{n} = 0.$$

$\square$

Question 7. Let $b_n = \sup_{k\geq n} |a_k|$ and $B_n = \sup_{k\geq n} \sqrt{|a_k|}$. Since $\sqrt{|a_k|} \leq B_n$ for $k \geq n$, we have $|a_k| \leq B_n^2$ for $k \geq n$ and so $b_n \leq B_n^2$ or $\sqrt{b_n} \leq B_n$. It follows that

$$\sqrt{\limsup_{n\to\infty} |a_n|} = \sqrt{\lim_{n\to\infty} b_n} = \lim_{n\to\infty} \sqrt{b_n} \leq \lim_{n\to\infty} B_n = \limsup_{n\to\infty} \sqrt{|a_n|}.$$

Conversely, since $|a_k| \leq b_n$ for $k \geq n$, we have $\sqrt{|a_k|} \leq \sqrt{b_n}$ for $k \geq n$ and so $B_n \leq \sqrt{b_n}$. It follows that

$$\limsup_{n\to\infty} \sqrt{|a_n|} = \lim_{n\to\infty} B_n \leq \lim_{n\to\infty} \sqrt{b_n} = \sqrt{\lim_{n\to\infty} b_n} = \sqrt{\limsup_{n\to\infty} |a_n|}$$

and hence the result.

$\square$

Question 8. Let $b_n = \sup\{a_n, a_{n+1}, \cdots\}$ and let $c_n = \inf\{a_n, a_{n+1}, \cdots\}$.
(i). Since $n_k \geq k$, we have

$$c_k = \inf\{a_k, a_{k+1}, \cdots\} \leq a_{n_k} \leq b_k = \sup\{a_k, a_{k+1}, \cdots\}$$

and so

$$C = \lim_{k\to\infty} c_k \leq \lim_{k\to\infty} a_{n_k} \leq \lim_{k\to\infty} b_k = B.$$

(ii). We construct a subsequence of $\{a_n\}$ as follows. Since

$$b_1 = \sup\{a_1, a_2, \cdots\},$$

$b_1 - 1$ is not an upper bound of $\{a_1, a_2, \cdots\}$ and so there exists a_{n_1} such that

$$a_{n_1} > b_1 - 1.$$

Since

$$b_{n_1+1} = \sup\{a_{n_1+1}, a_{n_1+2}, \cdots\},$$

$b_{n_1+1} - \frac{1}{2}$ is not an upper bound of $\{a_{n_1+1}, a_{n_1+2}, \cdots\}$ and so there exists a_{n_2} such that $n_2 > n_1$ and

$$a_{n_2} > b_{n_1+1} - \frac{1}{2}.$$

Now, by induction, suppose that we have constructed $a_{n_1}, a_{n_2}, \cdots, a_{n_k}$ such that $n_1 < n_2 < \cdot < n_k$ and

$$a_{n_s} > b_{n_{s-1}+1} - \frac{1}{s}$$

for $1 \leq s \leq k$. Since

$$b_{n_k+1} = \sup\{a_{n_k+1}, a_{n_k+2}, \cdots\},$$

$b_{n_k+1} - \frac{1}{k+1}$ is not an upper bound of $\{a_{n_k+1}, a_{n_k+2}, \cdots\}$ and so there exists $a_{n_{k+1}}$ such that $n_{k+1} > n_k$ and

$$a_{n_{k+1}} > b_{n_k+1} - \frac{1}{k+1}.$$

The induction is finished and so we obtain a subsequence $\{a_{n_1}, a_{n_2}, \cdots\}$ with the property that

$$a_{n_k} > b_{n_{k-1}+1} - \frac{1}{k}$$

for any k. Consider the inequality

$$b_{n_{k-1}+1} - \frac{1}{k} < a_{n_k} \leq b_{n_k}.$$

Since $\{b_n\}$ is convergent, we have

$$\lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty} b_{n_{k-1}+1} = \lim_{n\to\infty} b_n = B$$

and

$$\lim_{k\to\infty} (b_{n_{k-1}+1} - \frac{1}{k}) = B - 0 = B.$$

Thus, by the Squeeze theorem, we have

$$\lim_{k\to\infty} a_{n_k} = B = \overline{\lim_{n\to\infty}} a_n.$$

(iii). We construct a subsequence of $\{a_n\}$ as follows. Since

$$c_1 = \inf\{a_1, a_2, \cdots\},$$

$c_1 + 1$ is not a lower bound of $\{a_1, a_2, \cdots\}$ and so there exists a_{m_1} such that

$$a_{m_1} < c_1 + 1.$$

Since

$$c_{m_1+1} = \inf\{a_{m_1+1}, a_{m_1+2}, \cdots\},$$

$c_{m_1+1} + \dfrac{1}{2}$ is not a lower bound of $\{a_{m_1+1}, a_{m_1+2}, \cdots\}$ and so there exists a_{m_2} such that $m_2 > m_1$ and

$$a_{m_2} < c_{m_1+1} + \frac{1}{2}.$$

Now, by induction, suppose that we have constructed $a_{m_1}, a_{m_2}, \cdots, a_{m_k}$ such that $m_1 < m_2 < \cdot < m_k$ and

$$a_{m_s} < c_{m_{s-1}+1} + \frac{1}{s}$$

for $1 \leq s \leq k$. Since

$$c_{m_k+1} = \inf\{a_{m_k+1}, a_{m_k+2}, \cdots\},$$

$c_{m_k+1} + \dfrac{1}{k+1}$ is not a lower bound of $\{a_{m_k+1}, a_{m_k+2}, \cdots\}$ and so there exists $a_{m_{k+1}}$ such that $m_{k+1} > m_k$ and

$$a_{m_{k+1}} < c_{m_k+1} + \frac{1}{k+1}.$$

The induction is finished and so we obtain a subsequence $\{a_{m_1}, a_{m_2}, \cdots\}$ with the property that

$$a_{m_k} < c_{m_{k-1}+1} + \frac{1}{k}$$

for any k. Consider the inequality

$$c_{m_k} \leq a_{m_k} < c_{m_{k-1}+1} + \frac{1}{k}.$$

Since $\{c_n\}$ is convergent, we have

$$\lim_{k\to\infty} c_{m_k} = \lim_{k\to\infty} c_{m_{k-1}+1} = \lim_{n\to\infty} c_n = C$$

and

$$\lim_{k\to\infty} (c_{m_{k-1}+1} + \frac{1}{k}) = C + 0 = C.$$

Thus, by the Squeeze theorem, we have

$$\lim_{k\to\infty} a_{m_k} = C = \underline{\lim}_{n\to\infty} a_n.$$

The proof is finished. $\square$