

Question 1. Since $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, the limits $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} b_n$ exist. Thus the limits of $\{a_n + b_n\}$ and $\{a_n b_n\}$ exist and hence $\{a_n + b_n\}$ and $\{a_n b_n\}$ are also Cauchy sequences. \square

Question 2 (i). Let $a_n = \ln \frac{n+2}{n+3}$. Then the partial sum

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n = \ln \frac{1+2}{1+3} + \ln \frac{2+2}{2+3} + \ln \frac{3+2}{3+3} + \cdots + \ln \frac{n+2}{n+3} \\ &= \ln \frac{(1+2)(2+2)(3+2)(4+2) \cdots (n-1+2)(n+2)}{(1+3)(2+3)(3+3)(4+3) \cdots (n-1+3)(n+3)} \\ &= \ln \frac{3 \cdot 4 \cdot 5 \cdot 6 \cdots (n+1) \cdot (n+2)}{4 \cdot 5 \cdot 6 \cdot 7 \cdots (n+2) \cdot (n+3)} = \ln \frac{3}{n+3} = \ln 3 - \ln(n+3). \end{aligned}$$

Thus $\{S_n\}$ is divergent and so is the series $\sum_{n=1}^{\infty} \ln \frac{n+2}{n+3}$. \square

Question 2 (ii). Let $a_n = \frac{1}{n(n+2)}$. Observe that

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

The partial sum

$$\begin{aligned} S_n &= \frac{1}{1 \cdot (1+2)} + \frac{1}{2 \cdot (2+2)} + \cdots + \frac{1}{n \cdot (n+2)} \\ &= \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n+2} \right) \right] \\ &= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{1}{2} \left(\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)} \right). \end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n \rightarrow \infty} S_n = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

\square

Question 3 (a). Let $a_n = \frac{n^2 - 1}{2n^2 + n}$. Then $\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \neq 0$ and so the series $\sum_{n=1}^{\infty} \frac{n^2 - 1}{2n^2 + n}$ is divergent by the divergence test. \square

Question 3 (b). Let $a_n = \sin \frac{n\pi}{2}$. Then $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$ and so $\lim_{n \rightarrow \infty} a_n$ does not exist. Thus the series $\sum_{n=1}^{\infty} \sin \frac{n\pi}{2}$ is divergent by the divergence test. \square

Question 3 (c). Let $a_n = \frac{n^2 + 1 + \ln n}{n + n^3 + 4}$ and let $b_n = \frac{1}{n}$. Then

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{n + n^3 + 4}{n^2 + 1 + \ln n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} + 1 + \frac{4}{n^3}}{1 + \frac{1}{n^2} + \frac{\ln n}{n^2}} = \frac{0 + 1 + 0}{1 + 0 + 0} = 1.$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, so is $\sum_{n=1}^{\infty} \frac{n^2 + 1 + \ln n}{n + n^3 + 4}$ by the limit comparison test. \square

Question 3 (d). Observe that

$$\frac{3 + \sin n}{n^2} \leq \frac{4}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{4}{n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p -series, the positive series $\sum_{n=1}^{\infty} \frac{3 + \sin n}{n^2}$ is convergent by the comparison test. \square

Question 3 (e). Observe that

$$\frac{2^n + 3}{3^{n+1} - n} \leq \frac{2^n + 2^n}{3^{n+1}} = \frac{2^{n+1}}{3^{n+1}} = \left(\frac{2}{3}\right)^{n+1}$$

for $n \geq 2$. Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+1}$ is convergent by the geometric series, the positive series

$\sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n}$ is convergent by the comparison test. \square

Question 3 (f). Let $a_n = \frac{2}{n^{1+\frac{1}{n}}}$ and let $b_n = \frac{1}{n}$. Observe that

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{n}}}{2} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\sum_{n=1}^{\infty} \frac{2}{n^{1+\frac{1}{n}}}$ is divergent by the limit comparison test. \square

Question 3 (g). Observe that

$$\frac{4 + (-1)^n}{2n} \geq \frac{3}{2n}.$$

Since $\sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series

$\sum_{n=1}^{\infty} \frac{4 + (-1)^n}{2n}$ is divergent by the comparison test. \square

Question 3 (h). Observe that

$$\frac{1}{n(1 + \ln n)^p} = \frac{(1 + \ln n)^{-p}}{n} \geq \frac{1}{n}$$

for $p \leq 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series

$\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^p}$ is divergent for $p \leq 0$ by the comparison test. \square

Question 3 (i). Observe that

$$\frac{n}{n^2 + 1} \geq \frac{n}{n^2 + n^2} = \frac{n}{2n^2} = \frac{1}{2n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series

$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ is divergent by the comparison test. \square