Solutions to Tutorial 4

Question 1(a). Since $\sum_{n=1}^{\infty} a_n$ is convergent, we have $\lim_{n\to\infty} a_n = 0$ and so there exists a positive integer N such that $a_n = |a_n| = |a_n - 0| < 1$ for $n > N$. It follows

$$a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n$$

for $n > N$. By the comparison test, the positive series $\sum_{n=1}^{\infty} a_n^2$ is convergent. $\square$

Question 1 (b). Let $a_n = \frac{1}{n^2}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} \sqrt{a_n}$ is divergent by the p-series. $\square$

Question 2. Let $f(x) = \frac{1}{x(1+\ln x)}$. Then $f(x)$ is a positive, continuous monotone decreasing function over $[1, +\infty)$. Observe that

$$\int f(x)dx = \int \frac{1}{x(1+\ln x)}dx = \int \frac{1}{1+y}dy = \ln(1+y) + C = \ln(1+\ln x) + C,$$

where $y = \ln x$. The integral $\int_1^{\infty} f(x)dx$ is divergent and so is the series $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$ by the integral test. $\square$

Question 3 (a). Let $a_n = \frac{(3n)!}{6^n n!(2n)!}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[3(n+1)]!6^n n!(2n)!}{6^{n+1}(n+1)![2(n+1)]!(3n)!}$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{6(n+1)(2n+2)(2n+1)} = \frac{3\cdot 3\cdot 3}{6\cdot 2\cdot 2} = \frac{27}{24} > 1.$$

Thus the series $\sum_{n=1}^{\infty} \frac{(3n)!}{6^n n!(2n)!}$ is divergent by the ratio test. $\square$

Question 3 (b). Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} 2\left(1 - \frac{1}{n}\right)^n = \frac{2}{e} < 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent by the ratio test. $\square$

Question 4 (a). Let $a_n = \dfrac{5n^2 \cdot 3^n}{4^{n+4}}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{5^{\frac{1}{n}}(\sqrt[n]{n})^2 \cdot 3}{4 \cdot 4^{\frac{4}{n}}} = \frac{1 \cdot 1^2 \cdot 3}{4 \cdot 1} = \frac{3}{4} < 1.$$

Thus the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2 \cdot 3^n}{4^{n+4}}$ is convergent by the simplified root test. □

Question 4 (b). Let $a_n = \dfrac{3^{2n}}{5^n}\left(1 - \dfrac{1}{2n}\right)^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{3^2}{5}\left(1 + \frac{-\frac{1}{2}}{n}\right)^n = \frac{9}{5}e^{-\frac{1}{2}} = \frac{9}{5\sqrt{e}} > 1$$

because $e < \dfrac{9^2}{5^2} = \dfrac{81}{25} = 3.24$. Thus the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n}\left(1 - \frac{1}{2n}\right)^{n^2}$ is divergent by the simplified root test. □

Question 4 (c). Let a_n be the n-term in the series. Then $a_{2n-1} = \dfrac{1}{4^{2n-1}}$ and $a_{2n} = \dfrac{1}{5^{2n}}$. Thus

$$\sqrt[n]{a_n} = \begin{cases} \frac{1}{4} & \text{if} \quad \text{n is odd} \\ \frac{1}{5} & \text{if} \quad \text{n is even} \end{cases}$$

and so $\displaystyle\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{4} < 1$. Hence the series is convergent by the root test. □