2001/2002 Semester II MA2108 Advanced Calculus II

Solutions to Tutorial 5

Question 1 (a). Let $a_n = \sqrt{2n+2} - \sqrt{n}$. Then

$$a_n = \frac{(\sqrt{2n+2}-\sqrt{n})(\sqrt{2n+2}+\sqrt{n})}{\sqrt{2n+2}+\sqrt{n}} = \frac{2n+2-n}{\sqrt{2n+2}+\sqrt{n}} = \frac{n+2}{\sqrt{2n+2}+\sqrt{n}} \to \infty$$

as $n \to \infty$. The series $\displaystyle\sum_{n=1}^{\infty} \sqrt{2n+2} - \sqrt{n}$ is divergent by the divergence test. □

Question 1 (b). Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot \dfrac{2^n}{5^n}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1) \cdot 2^{n+1} \cdot n! \cdot 5^n}{(n+1)! \cdot 5^{n+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2^n} \\ &= \lim_{n\to\infty} \frac{(2n+1)\cdot 2}{(n+1)\cdot 5} = \frac{2 \cdot 2}{1 \cdot 5} = \frac{4}{5} < 1.\end{aligned}$$

Thus the series $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot \frac{2^n}{5^n}$ is convergent by the ratio test. □

Question 1 (c). Let $a_n = \dfrac{\ln n}{n^{1.2}}$ and let $b_n = \dfrac{1}{n^{1.1}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n \cdot n^{1.1}}{n^{1.2}} = \lim_{n\to\infty} \frac{\ln n}{n^{0.1}} = 0.$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is convergent by the p-series, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.2}}$ is convergent by the limit comparison test for the case $a_n << b_n$. □

Question 1 (d). Let $a_n = \left(\dfrac{n}{n+2}\right)^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+2}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)^n} = \frac{1}{e^2} < 1.$$

Thus the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$ is convergent by the simplified root test. □

Question 1 (e). Let $a_n = \dfrac{1}{n}$ and let $b_n = \dfrac{1}{(\ln n)^3}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^3}{n} = 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$ is divergent by the limit comparison test for the case $a_n << b_n$. ☐

Question 1 (f). Let $a_n = \left(\frac{4}{9} + \frac{n^3}{3^n}\right)^{\frac{n}{2}}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{4}{9} + \frac{n^3}{3^n}\right)^{\frac{1}{2}} = \left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{2}{3} < 1$$

and so the series is convergent by the simplified root test. ☐

Question 2 (i). Let $a_n = \frac{\ln n}{\sqrt{n}}$. Then $a_n \geq 0$. We show that a_n is eventually monotone decreasing. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{\frac{3}{2}}} \leq 0$$

for $x \geq e^2$ and so $\{a_n\}$ is monotone decreasing for $n \geq 9$. Since $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$ is convergent by the alternating series test. ☐

Question 2 (ii). Since $\left|(-1)^{n+1}\frac{\ln n}{\sqrt{n}}\right| \geq \frac{1}{n^{\frac{1}{2}}}$ for $n \geq 3$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent by the p-series, the series $\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{\ln n}{\sqrt{n}}\right|$ is divergent by the comparison test. By (i), the series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\ln n}{\sqrt{n}}$ is conditionally convergent. ☐

Question 3. Let $a_n = (-1)^n \frac{\cos n}{2^n}$. Then $|a_n| \leq \frac{1}{2^n}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent by the geometric series, the series $\sum_{n=1}^{\infty} \left|(-1)^n \frac{\cos n}{2^n}\right|$ is convergent by the comparison test and so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{2^n}$ is absolutely convergent. ☐

Question 4 (a). This series is conditionally convergent because it is convergent by the alternating series test and the series $\sum_{n=1}^{\infty} \left| (-1)^n \frac{3}{2n+1} \right|$ is divergent by the limit comparison test with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. ☐

Question 4 (b). Let $a_n = (-1)^n \frac{n}{4n+3}$. Then $\lim_{n\to\infty} a_{2n-1} = -\frac{1}{4}$ and $\lim_{n\to\infty} a_{2n} = \frac{1}{4}$. Thus the limit of $(-1)^n \frac{n}{4n+3}$ does not exist and so the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{4n+3}$ is divergent by the divergence test. ☐

Question 4 (c). Let $a_n = (-1)^n \left(\frac{1+2n}{3+4n} \right)^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1+2n}{3+4n} = \frac{2}{4} = \frac{1}{2} < 1.$$

Thus the positive series $\sum_{n=1}^{\infty} |a_n|$ is convergent by the simplified root test and so the series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1+2n}{3+4n} \right)^n$ is absolutely convergent. ☐

Question 4 (d). This series is conditionally convergent because it is convergent by the alternating series test but the series $\sum_{n=2}^{\infty} \left| (-1)^{n+1} \frac{1}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent by the integral test. ☐