Question 1. We use integral test estimation for solving this question. Let $f(x) = \dfrac{1}{x^5}$. From $\displaystyle\int_n^\infty f(x)dx = \frac{1}{-4}x^{-4}\Big|_n^{+\infty} = \frac{1}{4n^4} < 0.001$, we have $n > \sqrt[4]{\dfrac{1000}{4}} = 3.976$ or $n \geq 4$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.036$$

with the error less than 0.001. ☐

Question 2. We use alternating series test estimation for solving this question. Let $a_n = \dfrac{1}{n^5}$. From $a_{n+1} = \dfrac{1}{(n+1)^5} < 0.001$, we have $n + 1 > \sqrt[5]{1000}$ or $n \geq 3$ and so

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^5} \approx 1 - \frac{1}{2^5} + \frac{1}{3^5} \approx 0.9729$$

with error less than 0.001. ☐

3 (a). $F(x) = \lim_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^{nx} = (e^x)^x = e^{x^2}$.

3 (b). It does not converge pointwise because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

3 (c).

$$F(x) = \lim_{n\to\infty} \frac{x^{2n}}{1 + x^{2n}} = \begin{cases} \frac{0}{1+0} = 0 & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } \quad x = 1 \end{cases}$$

4 (a). The limiting function $F(x) = \lim_{n\to\infty} \dfrac{n^2}{n^2 + x^2} = \lim_{n\to\infty} \dfrac{1}{1 + \frac{x^2}{n^2}} = 1$.

$$0 \leq T_n = \sup_{0\leq x\leq 1} |F_n(x) - F(x)| = \sup_{0\leq x\leq 1} |\frac{n^2}{n^2 + x^2} - 1| = \sup_{0\leq x\leq 1} \frac{x^2}{n^2 + x^2} \leq \frac{1}{n^2}.$$

Since $\lim_{n\to\infty} \dfrac{1}{n^2} = \lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so $\{F_n\}$ converges uniformly on $[0, 1]$.

4 (b). The limiting function $F(x) = \lim_{n\to\infty} x^n(1 - x) = 0$ for $0 \leq x \leq 1$. Observe

$$T_n = \sup_{0\leq x\leq 1} |F_n(x) - F(x)| = \sup_{0\leq x\leq 1} x^n(1 - x).$$

Let $g(x) = x^n(1 - x)$. Then $g'(x) = nx^{n-1}(1 - x) - x^n = x^{n-1}[n - (n + 1)x]$. From $g'(x) = 0$, we have $x = 0$ or $\frac{n}{n+1}$. Since $g'(x) \geq 0$ for $0 \leq x \leq \dfrac{n}{n+1}$ and $g'(x) \leq 0$ for

$\frac{n}{n+1} \leq x \leq 1$, $\sup\limits_{0\leq x\leq 1} g(x) = \max\{g(x)|0 \leq x \leq 1\} = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$ and so

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0.$$

Thus $\{F_n\}$ converges uniformly on $[0, 1]$.

4 (c). The limiting function $f(x) = \lim\limits_{n\to\infty} \frac{n \ln x}{x^n} = 0$ for $1 \leq x < \infty$. Observe that

$$T_n = \sup_{x\geq 1} |f_n(x) - f(x)| = \sup_{x\geq 1} \left|\frac{n \ln x}{x^n} - 0\right| = \sup_{x\geq 1} \frac{n \ln x}{x^n}.$$

Let $g(x) = \frac{n \ln x}{x^n}$. From

$$g'(x) = \left(n \ln x \cdot x^{-n}\right)' = n\frac{1}{x}x^{-n} - n^2 \ln x \cdot x^{-n-1} = \frac{n - n^2 \ln x}{x^{n+1}} = 0,$$

we have $x = e^{\frac{1}{n}}$. Since $g'(x) \geq 0$ for $1 \leq x \leq e^{\frac{1}{n}}$ and $g'(x) \leq 0$ for $x \geq e^{\frac{1}{n}}$, we have

$$T_n = \sup_{x\geq 1} g(x) = \max\{g(x)|x \geq 1\} = \frac{n \ln e^{\frac{1}{n}}}{\left(e^{\frac{1}{n}}\right)^n} = \frac{1}{e}$$

and so $\lim\limits_{n\to\infty} T_n = \frac{1}{e} \neq 0$. Thus $\{f_n\}$ does not converge uniformly on $[1, +\infty)$.

4 (d). The limiting function $f(x) = \lim\limits_{n\to\infty} \frac{n \ln x \cos nx}{x^n} = 0$ for $x \geq 4$. Observe that

$$0 \leq T_n = \sup_{x\geq 4} |f_n(x) - f(x)| = \sup_{x\geq 4} \frac{n \ln x \cdot |\cos nx|}{x^n} \leq \sup_{x\geq 4} \frac{n \ln x}{x^n} = g(4) = \frac{n \ln 4}{4^n},$$

where $g(x) = \frac{n \ln x}{x^n}$ is monotone decreasing on $[4, +\infty)$. Since $\lim\limits_{n\to\infty} \frac{n \ln 4}{4^n} = \lim\limits_{n\to\infty} 0 = 0$, we have $\lim\limits_{n\to\infty} T_n = 0$ by the Squeeze theorem and so $\{f_n\}$ converges uniformly on $[4, +\infty)$.

4 (e). The limiting function $F(x) = \lim\limits_{n\to\infty} F_n(x) = \lim\limits_{n\to\infty} \frac{n^2}{n^2 + x^2} = 1$. Observe that

$$T_n = \sup_{x\geq 0} |F_n(x) - F(x)| = \sup_{x\geq 0} \left|\frac{n^2}{n^2 + x^2} - 1\right| = \sup_{x\geq 0} \frac{x^2}{n^2 + x^2}.$$

Let $g(x) = \frac{x^2}{n^2 + x^2}$. Since

$$g'(x) = \frac{2x(n^2 + x^2) - x^2 \cdot 2x}{(n^2 + x^2)^2} = \frac{2xn^2}{(n^2 + x^2)^2} \geq 0$$

for $x \geq 0$, the function $g(x)$ is monotone increasing and so

$$T_n = \sup_{x\geq 0} g(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^2}{n^2+x^2} = \lim_{x\to\infty} \frac{2x}{2x} = 1.$$

Thus $\lim\limits_{n\to\infty} T_n = 1 \neq 0$ and so $\{F_n(x)\}$ does not converge uniformly on $[0, +\infty)$.

5. For $\epsilon = 1$, since $F_n(x)$ converges uniformly to $F(x)$ on I, there exists N such that

$$|F_n(x) - F(x)| < 1$$

for all $x \in I$ and $n > N$. Thus $|F_{N+1}(x) - F(x)| < 1$ or

$$F_{N+1}(x) - 1 < F(x) < F_{N+1}(x) + 1$$

for all $x \in I$ and so

$$|F(x)| < \max\{|F_{N+1}(x) + 1|, |F_{N+1}(x) - 1|\} \leq |F_{N+1}(x)| + 1 \leq M_{N+1} + 1$$

for all $x \in I$. Let $M = M_{N+1} + 1$. Then $|F(x)| \leq M$ for all $x \in I$.