Question 1 (i). Let $F_n(x) = \dfrac{n+e^x}{n+x^2}$. Then the limiting function

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{n+e^x}{n+x^2} = \lim_{n\to\infty} \frac{1+\frac{e^x}{n}}{1+\frac{x^2}{n}} = 1$$

and

$$0 \le T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} \left| \frac{n+e^x}{n+x^2} - 1 \right|$$

$$= \sup_{0\le x\le 1} \frac{|e^x - x^2|}{n^2+x^2} \le \sup_{0\le x\le 1} \frac{e^x + x^2}{n^2+x^2} \le \frac{e+1}{n^2}$$

Since $\lim_{n\to\infty} \dfrac{e+1}{n^2} = \lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem. Thus $\{F_n\}$ converges uniformly to $F(x)$ and so

$$\lim_{n\to\infty} \int_0^1 \frac{n+e^x}{n+x^2} dx = \int_0^1 \lim_{n\to\infty} F_n(x) dx = \int_0^1 1 dx = 1.$$

□

Question 1 (ii). Let $F_n(x) = \left(\dfrac{x^2+1}{8}\right)^n \sin nx$. Then

$$0 \le |F_n(x)| = \left| \left(\frac{x^2+1}{8}\right)^n \sin nx \right| \le \left(\frac{2^2+1}{8}\right)^n = \left(\frac{5}{8}\right)^n$$

for $1 \le x \le 2$. Since $\lim_{n\to\infty} \left(\dfrac{5}{8}\right)^n = \lim_{n\to\infty} 0 = 0$, the limiting function $F(x) = 0$ for $1 \le x \le 2$ and

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \sup_{1\le x\le 2} |F_n(x) - F(x)| = \lim_{n\to\infty} \sup_{1\le x\le 2} |F_n(x)| = 0$$

Thus $\{F_n\}$ converges uniformly to $F(x)$ and so

$$\lim_{n\to\infty} \int_1^2 \left(\frac{x^2+1}{8}\right)^n \sin nx dx = \int_1^2 \lim_{n\to\infty} F_n(x) dx = \int_1^2 0 dx = 0.$$

□

Question 2 (i). Since

$$\left| \frac{\cos nx}{n^2+x^2} \right| \le \frac{1}{n^2}$$

and the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^2+x^2}$ converges uniformly on $(-\infty, +\infty)$ by the Weierstrass M-test. □

Question 2 (ii). Since
$$\left|\frac{1}{1+n^3x^2}\right| \leq \frac{1}{1+2^2n^3} \leq \frac{1}{n^3}$$
for $x \geq 2$ and the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty}\frac{1}{1+n^3x^2}$ converges uniformly on $[2,+\infty)$ by the Weierstrass M-test. □

Question 2 (iii). Let $f_n(x) = \frac{xe^{-x}}{n^2}$ for $x \in (0,+\infty)$. Then
$$f_n'(x) = \frac{1}{n^2}\left(e^{-nx} - xne^{-nx}\right) = \frac{(1-nx)e^{-nx}}{n^2}.$$
Thus $f_n(x)$ is monotone increasing for $0 \leq x \leq \frac{1}{n}$ and monotone decreasing for $x \geq \frac{1}{n}$. It follows that
$$|f_n(x)| \leq f_n(\frac{1}{n}) = \frac{\frac{1}{n}e^{-n\cdot\frac{1}{n}}}{n^2} = \frac{e^{-1}}{n^3}$$
for $x \in (0,+\infty)$. Since the series $\sum_{n=1}^{\infty}\frac{e^{-1}}{n^3} = e^{-1}\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent by the p-series, the series of functions $\sum_{n=1}^{\infty}\frac{xe^{-nx}}{n^2}$ converges uniformly on $(0,+\infty)$ by the Weierstrass M-test. □

Question 3. Let $F_n(x) = \sum_{k=1}^{n} f_k(x)$ and let $G_n(x) = \sum_{k=1}^{n} g_k(x)$. Since the series of functions $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on I, the sequence of functions $\{G_n(x)\}$ converges uniformly on I. By the Cauchy Criterion, for any $\epsilon > 0$, there exists such that, for all $x \in I$ and all $n > m > N$,
$$|G_n(x) - G_m(x)| < \epsilon \quad \Rightarrow \quad \left|\sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{m} g_k(x)\right| < \epsilon$$
$$\Rightarrow \quad \sum_{k=m+1}^{n} g_k(x) = \left|\sum_{k=m+1}^{n} g_k(x)\right| < \epsilon \qquad \text{because } g_k(x) \geq |f_k(x)| \geq 0.$$
For all $x \in I$ and $n > m > N$, we have
$$|F_n(x) - F_m(x)| = \left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| = \left|\sum_{k=m+1}^{n} f_k(x)\right|$$

$$\leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} g_k(x) < \epsilon.$$

Thus, by the Cauchy Criterion, the sequence of functions $\{F_n(x)\}$ converges uniformly and so does the series of functions $\sum_{n=1}^{\infty} f_n(x)$.

□