

1. Let $f_k(x) = \frac{(-1)^k x^k}{1+x^{2k}}$ for $0 < x < \frac{2}{3}$. Since

$$|f_k(x)| = \left| \frac{(-1)^k x^k}{1+x^{2k}} \right| \leq \frac{\left(\frac{2}{3}\right)^k}{1} = \left(\frac{2}{3}\right)^k$$

for $0 < x < \frac{2}{3}$ and the series $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ converges by the geometric series, the series of functions $\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{1+x^{2k}}$ converges uniformly on $\left(0, \frac{2}{3}\right)$ by the Weierstrass M -test.

Thus the function $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^k x^k}{1+x^{2k}}$ is continuous on the interval $\left(0, \frac{2}{3}\right)$.

2. Let $f_n(x) = \frac{x^n(1-x^2)}{\sqrt{1+x}}$ for $0 \leq x \leq \frac{1}{2}$. Since

$$|f_n(x)| = \frac{|x|^n |1-x^2|}{\sqrt{1+x}} \leq \frac{\left(\frac{1}{2}\right)^n \cdot 1}{1} = \left(\frac{1}{2}\right)^n$$

for $0 \leq x \leq \frac{1}{2}$ and the series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent by the geometric series, the series of functions $\sum_{n=0}^{\infty} \frac{x^n(1-x^2)}{\sqrt{1+x}}$ converges uniformly on $\left[0, \frac{1}{2}\right]$ by the M -test. Note

that each $f_n(x)$ is Riemann integrable on $\left[0, \frac{1}{2}\right]$. We have

$$\begin{aligned} \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} \frac{x^n(1-x^2)}{\sqrt{1+x}} dx &= \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{x^n(1-x^2)}{\sqrt{1+x}} dx = \int_0^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} x^n \right) \cdot \frac{1-x^2}{\sqrt{1+x}} dx \\ &= \int_0^{\frac{1}{2}} \frac{1}{1-x} \cdot \frac{1-x^2}{\sqrt{1+x}} dx = \int_0^{\frac{1}{2}} \sqrt{1+x} dx = \frac{2}{3} (1+x)^{\frac{3}{2}} \Big|_0^{\frac{1}{2}} = \frac{2}{3} \left[\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1 \right] = \sqrt{\frac{3}{2}} - \frac{2}{3}. \end{aligned}$$

3(i). Since

$$|a_k \sin kx| \leq |a_k|$$

for $x \in (-\infty, +\infty)$ and $\sum_{k=1}^{\infty} |a_k|$ is convergent, the series of functions $\sum_{k=1}^{\infty} a_k \sin kx$ converges uniformly on $(-\infty, +\infty)$ by the M -test.

3 (ii). By (i), the series of functions $\sum_{k=1}^{\infty} a_k \sin kx$ converges uniformly on $[0, 2\pi]$.

Since each $a_k \sin kx$ is Riemann integrable, we have

$$\int_0^{2\pi} \sum_{k=1}^{\infty} a_k \sin kx dx = \sum_{k=1}^{\infty} \int_0^{2\pi} a_k \sin kx dx = \sum_{k=1}^{\infty} 0 = 0.$$

4. From $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any x , we have

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}.$$

Since

$$\left| \frac{(-1)^n x^{3n}}{n!} \right| \leq \frac{1}{n!}$$

for $0 \leq x \leq 1$ and the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent by the ratio test, the series of

functions $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$ converges uniformly to e^{-x^3} on $[0, 1]$ by the M -test. Note

that each $\frac{(-1)^n x^{3n}}{n!}$ is Riemann integrable. Thus

$$\int_0^1 e^{-x^3} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(3n+1)}.$$

Let $a_n = \frac{1}{n!(3n+1)}$. Then the sequence $\{a_n\}$ is positive, monotone decreasing and

$\lim_{n \rightarrow \infty} a_n = 0$. By applying the alternating test estimation, from $a_{n+1} < 0.001$ or $(n+1)!(3n+4) \geq 1000$, we have $n \geq 4$ and so

$$\int_0^1 e^{-x^3} dx \approx 1 - \frac{1}{1! \cdot 4} + \frac{1}{2! \cdot 7} - \frac{1}{3! \cdot 10} + \frac{1}{4! \cdot 13} = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312}$$

with error less than 0.001.

5. Let $f_n(x) = \frac{\cos^n x}{n^3}$. Given any point x_0 in $(-\infty, +\infty)$, let a and b be any numbers such that $a < x_0 < b$. Then

(1). Each $f'_n(x) = \frac{-\cos^{n-1} x \sin x}{n^2}$ is continuous on $[a, b]$.

(2). The series of functions $\sum_{n=1}^{\infty} f_n(x)$ absolutely converges on $[a, b]$ by the comparison test because $\left| \frac{\cos^n x}{n^3} \right| \leq \frac{1}{n^3}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the p -series. So it converges pointwise on $[a, b]$.

(3). The series of functions $\sum_{n=1}^{\infty} f'_n(x)$ converges uniformly on $[a, b]$ by the M -test because $\left| \frac{-\cos^{n-1} x \sin x}{n^2} \right| \leq \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p -series.

Thus the function $f(x) = \sum_{n=1}^{\infty} \frac{\cos^n x}{n^3}$ is differentiable on $[a, b]$ and so at x_0 . Since x_0 is any given point in $(-\infty, +\infty)$, the function $f(x)$ is differentiable on $(-\infty, +\infty)$.
6 i).

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\limsup \sqrt[n]{\left(1 + \frac{3}{n}\right)^{n^2}}} = \frac{1}{\lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right)^n} = \frac{1}{e^3}.$$

6 ii)

$$R = \frac{1}{\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{|-3|^{n+1} \cdot n!}{(n+1)! \cdot |-3|^n}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{3}{n+1}} = +\infty.$$

6 iii) Since $a_n = \left(\frac{1}{5}\right)^n$ if n is odd and $\left(\frac{1}{6}\right)^n$ if n is even, $\sqrt[n]{|a_n|} = \frac{1}{5}$ if n is odd and $\frac{1}{6}$ if n is even. Thus

$$b_n = \sup_{k \geq n} \sqrt[k]{|a_k|} = \frac{1}{5}$$

and so $\limsup \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} b_n = \frac{1}{5}$. It follows that

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = 5.$$

6 iv) Observe that

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{3^n}{n^2} \cdot \left(x - \frac{2}{3}\right)^n.$$

Thus

$$R = \frac{1}{\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{3^{n+1} \cdot n^2}{(n+1)^2 \cdot 3^n}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{3}{\left(1 + \frac{1}{n}\right)^2}} = \frac{1}{3}.$$