1 i).

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{|-2|^{n+1}\cdot n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}}\cdot|-2|^n}} = \frac{1}{\lim\limits_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}} = \frac{1}{2}.$$

Consider the ending points $x = x_0 \pm R = \pm\frac{1}{2}$. The series $\sum_{n=1}^{\infty} \frac{[-2\cdot(-\frac{1}{2})]^n}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is convergent by the p-series and the series $\sum_{n=1}^{\infty} \frac{(-2\cdot\frac{1}{2})^n}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$ is convergent by the alternating series test. Thus the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$.

1 ii).

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{3^{n+1}\cdot(n+1)}{(n+2)\cdot 3^n}} = \frac{1}{\lim\limits_{n\to\infty} \frac{3\cdot\left(1+\frac{1}{n}\right)}{1+\frac{2}{n}}} = \frac{1}{3}.$$

Consider the ending points $x = x_0 \pm R = 2 \pm \frac{1}{3}$. When $x = 2 - \frac{1}{3}$,

$$\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n+1} = \sum_{n=1}^{\infty} \frac{3^n\left(-\frac{1}{3}\right)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$

is convergent by the alternating series test. When $x = 2 + \frac{1}{3}$,

$$\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n+1} = \sum_{n=1}^{\infty} \frac{3^n\left(\frac{1}{3}\right)^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$$

is divergent by the p-series. Thus the interval of convergence is $\left[2-\frac{1}{3}, 2+\frac{1}{3}\right)$.

1 iii). Observe that

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n}\cdot\left(x-\frac{1}{3}\right)^n.$$

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\frac{|-3|^n}{n}}} = \frac{1}{\lim\limits_{n\to\infty} \frac{3}{\sqrt[n]{n}}} = \frac{1}{3}.$$

Consider the ending points $x = x_0 \pm R = \frac{1}{3} \pm \frac{1}{3}$. When $x = 0$,

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent by the p-series. When $x = \frac{2}{3}$,

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is convergent by the alternating series test. Thus the interval of convergence is $\left(0, \dfrac{2}{3}\right]$.

2. Given any $x \in (-1, 1)$. If $x = 0$, clearly $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$. For $x \neq 0$, we may assume that $0 < x < 1$. Let $f_n(t) = (-1)^n t^n$. Since $|f_n(t)| \leq x^n$ for $0 \leq t \leq x$ and $\displaystyle\sum_{n=0}^{\infty} x^n$ is convergent by the geometric series, the series of functions $\sum_{n=0}^{\infty} f_n(t)$ converges uniformly on $[0, x]$ and so

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^n dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n dt$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

3. Given any x, choose $b > 0$ such that $-b < x < b$. Let $f_n(t) = \dfrac{t^n}{n!}$ and let

$$g(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Then

1) $f_n'(t)$ exists and is continuous on $[-b, b]$;
2) the series of functions $\displaystyle\sum_{n=0}^{\infty} f_n(t)$ converges pointwise on $[-b, b]$ and
3) the series of functions $\displaystyle\sum_{n=0}^{\infty} f_n'(t)$ converges uniformly on $[-b, b]$, by the Weierstrass M-test, because

$$|f_n'(t)| = \left|\frac{t^{n-1}}{(n-1)!}\right| \leq \frac{b^{n-1}}{(n-1)!}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{b^{n-1}}{(n-1)!}$ converges.

Thus

$$g'(t) = \sum_{n=0}^{\infty} \left(\frac{t^n}{n!}\right)' = 0 + 1 + t + \frac{t^2}{2!} + \cdots = g(t).$$

Let $y(t) = g(t)$. We obtain the differential equation

$$\frac{dy}{dt} = y \quad \Rightarrow \quad \int \frac{dy}{y} = \int dt$$

$$\Rightarrow \ln|y| = t + k \quad \Rightarrow \quad y = Ce^t,$$

where $C = \pm e^k$. Let $t = 0$, we have

$$C \cdot e^0 = y(0) = 1 + 0 + 0 + \cdots = 1$$

and so $y = e^t$ on $[-b, b]$. It follows $y(x) = e^x$ because $x \in [-b, b]$. Since x is arbitrarily given,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for all $x \in (-\infty, +\infty)$.

4.

$$\int_0^{0.1} \frac{1}{\sqrt{(1+x^4)}} dx = \int_0^{\frac{1}{10}} \left(1+x^4\right)^{-\frac{1}{2}} dx = \int_0^{\frac{1}{10}} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k} dx$$

$$= \sum_{k=0}^{\infty} \int_0^{\frac{1}{10}} \binom{-\frac{1}{2}}{k} x^{4k} dx = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{1}{(4k+1) \cdot 10^{4k+1}},$$

where $\int_0^{\frac{1}{5}} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \int_0^{\frac{1}{5}}$ because the series of functions $\sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k}$ converges uniformly on $\left[0, \frac{1}{10}\right]$. Observe that

$$\binom{-\frac{1}{2}}{k} = (-1)^k \frac{(2k-1)!!}{2^k \cdot k!}.$$

The series

$$\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{1}{(4n+1) \cdot 10^{4k+1}} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{2^k \cdot k! \cdot (4k+1) 10^{4k+1}}$$

is an alternating series. Let $a_k = \dfrac{(2k-1)!!}{2^k \cdot k! \cdot (4k+1) 10^{4k+1}}$. Then

(1). $a_k > 0$;
(2). $\{a_k\}$ monotone decreasing because

$$\frac{a_{k+1}}{a_k} = \frac{(2k+1)!! \cdot 2^k \cdot k! \cdot (4k+1) 10^{4k+1}}{2^{k+1} \cdot (k+1)! \cdot (4k+5) 10^{4k+5} \cdot (2k-1)!!} = \frac{(2k+1) \cdot (4k+1)}{2 \cdot (k+1) \cdot (4k+5) \cdot 10^4} < 1;$$

(3). By the binomial series $\sum_{k=0}^{\infty} (-1)^k a_k$ converges. Thus $\lim_{k \to \infty} (-1)^k a_k = 0$ and so $\lim_{k \to \infty} a_k = 0$.

From

$$\frac{(2k+1)!!}{2^{k+1}\cdot(k+1)!\cdot(4n+5)\cdot 10^{4k+5}} < 10^{-8},$$

we have $k \geq 1$ and so

$$\int_0^{0.1} \frac{1}{\sqrt{(1+x^4)}}dx \approx \frac{1}{10} - \frac{1}{2\cdot 5\cdot 10^5} = \frac{1}{10} - \frac{1}{10^6} = 0.099999$$

with error $< 10^{-8}$.

5. From

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(x) = \sum_{k=0}^{\infty} \binom{\frac{1}{5}}{k} x^{3k},$$

we have $\dfrac{f^{(30)}(0)}{30!} = \dbinom{\frac{1}{5}}{10}$ or

$$f^{(30)}(0) = 30!\binom{\frac{1}{5}}{10} = \frac{30!\cdot\frac{1}{5}\cdot(\frac{1}{5}-1)\cdots(\frac{1}{5}-9)}{10!}.$$

6.

$$\begin{aligned}\sqrt[3]{9} = (8+1)^{\frac{1}{3}} = 2\left(1+\frac{1}{8}\right)^{\frac{1}{3}} &= 2\sum_{k=0}^{\infty}\binom{\frac{1}{3}}{k}\frac{1}{8^k} \\ &= 2 + \sum_{k=1}^{\infty}\frac{\frac{1}{3}\cdot(\frac{1}{3}-1)\cdots(\frac{1}{3}-k+1)}{k!\cdot 2^{3k-1}} \\ &= 2 + \frac{1}{12} + \sum_{k=2}^{\infty}(-1)^{k-1}\frac{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3}-1)}{k!\cdot 2^{3k-1}}\end{aligned}$$

Let $a_k = \dfrac{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3}-1)}{k!\cdot 2^{3k-1}}$ for $k \geq 2$. Then

1) $a_k > 0$, that is, the series is alternating.

2) Since

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3})}{(k+1)!\cdot 2^{3k+2}}\cdot\frac{k!\cdot 2^{3k-1}}{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3}-1)} \\ &= \frac{k-\frac{1}{3}}{(k+1)\cdot 8} \leq 1,\end{aligned}$$

$a_{k+1} \leq a_k$ or $\{a_k\}$ is monotone deceasing.

3) Observe that

$$0 \leq a_k = \frac{\frac{1}{3}\cdot(1-\frac{1}{3})\cdots(k-\frac{1}{3}-1)}{k!\cdot 2^{3k-1}} < \frac{\frac{1}{3}\cdot 1\cdot 2\cdots(k-1)}{k!\cdot 2^{3k-1}} = \frac{1}{3\cdot k\cdot 2^{3k-1}}$$

for $k \geq 2$. Since $\lim_{k\to\infty}\dfrac{1}{3\cdot k\cdot 2^{3k-1}} = 0$, $\lim_{k\to\infty} a_k = 0$ by the Squeeze theorem.

Thus we can apply the alternating series estimation. From

$$a_{k+1} = \frac{\frac{1}{3} \cdot (1 - \frac{1}{3}) \cdots (k - \frac{1}{3})}{(k+1)! \cdot 2^{3k+2}} < 10^{-4},$$

we have $k \geq 3$. Thus

$$\sqrt[3]{9} \approx 2 + \frac{1}{12} - \frac{1}{288} + \frac{5}{20746} = 2.07988533521312364950631280098447$$

with an error of magnitude less than 10^{-4}.