Lecture Notes On Advanced Calculus II

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Contents

CHAPTER 1

Sequences

1. Sequences

A *sequence* is an ordered list of numbers. For example,

$$1, 2, 3, 4, 5, 6$$

The order of the sequence is important. For example,

$$2, 1, 4, 3, 6, 5$$

is different from above sequence. An *infinite* sequence is a list which does not end. For example,

$$1, 1/2, 1/3, 1/4, 1/5, \cdots$$

We are going to study infinite sequences. We denote by $\{a_n\}$ the sequence

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

Example 1.1. *Here are some examples of infinite sequences.*

(1). $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$

(2). $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \cdots$

(3). $1, -2, 3, -4, 5, \cdots$

Can you find a formula for each of the above sequences?
Answer: (1). $a_n = 1/n$. (2). $a_n = 1/3^n$. (3). $(-1)^{n-1}n$.

2. Limits of Sequences

Definition 2.1. *The limit of $\{a_n\}$ is A, and is written as*

$$\lim_{n\to\infty} a_n = A,$$

if for any $\epsilon > 0$, there is a natural number N such that for every $n > N$, we have

$$|a_n - A| < \epsilon.$$

Remark. 1. Some sequences do not satisfy the above. We call such sequences *divergent.*
2. Sequences which satisfy the above definition, i.e. A exists and is finite, are called *convergent* sequences.

Example 2.2. *Prove the following limits by using $\epsilon - N$ definition*

1) $\lim_{n\to\infty} \frac{1}{n} = 0.$

2) $\lim\limits_{n\to\infty} \sqrt{\dfrac{n^2}{n^2+1}} = 1.$

3) $\lim\limits_{n\to\infty} \left(\dfrac{3}{4}\right)^n = 0.$

(1). Given any $\epsilon > 0$, we want to find N such that $\left|\dfrac{1}{n} - 0\right| < \epsilon$ for $n > N$, i.e., $n > \dfrac{1}{\epsilon}$ for $n > N$. Choose N to be the smallest integer such that $N \geq \dfrac{1}{\epsilon}$. ($N$ is found now!) When $n > N$, then $n > N \geq \dfrac{1}{\epsilon}$ or $\left|\dfrac{1}{n} - 0\right| < \epsilon$. Thus $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

(2). Given any $\epsilon > 0$, we want to find N such that $\left|\sqrt{\dfrac{n^2}{n^2+1}} - 1\right| < \epsilon$ for $n > N$. Now

$$\left|\sqrt{\frac{n^2}{n^2+1}} - 1\right| < \epsilon \Leftrightarrow \left|\frac{n}{\sqrt{n^2+1}} - 1\right| < \epsilon \Leftrightarrow \left|\frac{n - \sqrt{n^2+1}}{\sqrt{n^2+1}}\right| < \epsilon$$

$$\Leftrightarrow \left|\frac{n^2 - (n^2+1)}{\sqrt{n^2+1}(n + \sqrt{n^2+1})}\right| < \epsilon \Leftrightarrow \frac{1}{\sqrt{n^2+1}(n + \sqrt{n^2+1})} < \epsilon$$

$$\Leftrightarrow \sqrt{n^2+1}(n + \sqrt{n^2+1}) > \frac{1}{\epsilon}$$

Choose N to be the smallest integer such that $N \geq \frac{1}{\epsilon}$. Then, for $n > N$,

$$\sqrt{n^2+1}(n + \sqrt{n^2+1}) > \sqrt{N^2+1}(N + \sqrt{N^2+1}) > N \geq \frac{1}{\epsilon}$$

or $\left|\sqrt{\dfrac{n^2}{n^2+1}} - 1\right| < \epsilon$. Thus N is found and hence the result.

(3). Given any $\epsilon > 0$, we want to find N such that $\left|\left(\dfrac{3}{4}\right)^n - 0\right| < \epsilon$ for $n > N$. Observe that

$$\left(\frac{3}{4}\right)^n < \epsilon \Leftrightarrow n \ln\left(\frac{3}{4}\right) < \ln(\epsilon) \Leftrightarrow n > \frac{\ln(\epsilon)}{\ln(3/4)}$$

(**Note.** $\ln(3/4) < 0$!!) Choose N to be the smallest positive integer such that $N \geq \dfrac{\ln(\epsilon)}{\ln(3/4)}$. When $n > N$, then

$$n > N \geq \frac{\ln(\epsilon)}{\ln(3/4)}$$

or $\left|\left(\dfrac{3}{4}\right)^n - 0\right| < \epsilon$. The proof is finished.

THEOREM 2.3. *If $\{a_n\}$ has a limit, then the limit is unique.*

PROOF. Let A and B be limits of $\{a_n\}$. Suppose that $A \neq B$. Choose $\epsilon = \dfrac{|A-B|}{2}$. Then $\epsilon > 0$ because $A \neq B$. By definition, there exists N_1 and N_2 such that $|a_n - A| < \epsilon$ for $n > N_1$ and $|a_n - B| < \epsilon$ for $n > N_2$. For $n > \max\{N_1, N_2\}$, we have

$$|A-B| = |(A-a_n)+(a_n-B)| \le |A-a_n| + |a_n-B| < 2\epsilon = 2\frac{|A-B|}{2} = |A-B|,$$

which is a contradiction. Thus $A = B$. □

THEOREM 2.4 (Squeeze or Sandwich Theorem). *Given* 3 *sequences*

$$\{a_n\}, \{b_n\}, \{c_n\}$$

such that

(i) $a_n \le b_n \le c_n$ *for every* n *and*

(ii) $\lim\limits_{n\to\infty} a_n = A = \lim\limits_{n\to\infty} c_n$,

then $\lim\limits_{n\to\infty} b_n = A$.

PROOF. For any $\epsilon > 0$, there exists N_1 and N_2 such that $|c_n - A| < \epsilon$ for $n > N_1$ and $|a_n - A| < \epsilon$ for $n > N_2$. Let $N = \max\{N_1, N_2\}$. For $n > N$, we have

$$-\epsilon < c_n - A < \epsilon \quad \text{and} \quad -\epsilon < a_n - A < \epsilon$$

$$A-\epsilon < c_n < A+\epsilon \quad \text{and} \quad A-\epsilon < a_n < A+\epsilon.$$

Thus

$$A-\epsilon < a_n \le b_n \le c_n < A+\epsilon \quad \text{or} \quad |b_n - A| < \epsilon.$$

By definition, we have $\lim\limits_{n\to\infty} b_n = A$ and hence the result. □

Remark. The above theorem is still applicable if the inequality

$$a_n \le b_n \le c_n$$

is true “eventually”.

EXAMPLE 2.5. *Find limits*

1) $\lim\limits_{n\to\infty} \dfrac{1+\sin n}{n}$.

2) $\left(\dfrac{3n-1}{4n+1}\right)^n$.

(1). Since

$$0 \le \frac{1+\sin n}{n} \le \frac{2}{n}$$

and $\lim\limits_{n\to\infty} \dfrac{2}{n} = \lim\limits_{n\to\infty} 0 = 0$, we have

$$\lim_{n\to\infty} \frac{1+\sin n}{n} = 0.$$

(2). Since

$$0 \le \left(\frac{3n-1}{4n+1}\right)^n \le \left(\frac{3}{4}\right)^n$$

and $\lim\limits_{n\to\infty}\left(\frac{3}{4}\right)^n = \lim\limits_{n\to\infty} 0 = 0$, we have

$$\lim_{n\to\infty}\left(\frac{3n-1}{4n+1}\right)^n = 0.$$

3. Sequences which tend to ∞

DEFINITION 3.1. $\{a_n\}$ *tends to* $+\infty$ *if for each positive number* k*, there is an* N *such that*

$$a_n > k \quad \textit{for all } n > N.$$

Remark. For such sequences, we write as $a_n \to +\infty$ as $n \to \infty$ or

$$\lim_{n\to\infty} a_n = +\infty.$$

EXAMPLE 3.2. The following sequences tend to $+\infty$

1) $a_n = \sqrt{\ln n}$.
2) $a_n = (3/2)^n$.

The sequences $-\ln n$, $-n^2$ and etc then tend to $-\infty$.

THEOREM 3.3 (Reciprocal Rule). *Consider a sequence* $\{a_n\}$.

(i) *If* $a_n > 0$ *for all* n *and* $\lim_{n\to\infty} \frac{1}{a_n} = 0$*, then*

$$\lim_{n\to\infty} a_n = +\infty.$$

(ii) *If* $\lim_{n\to\infty} a_n = \pm\infty$*, then* $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

PROOF. We only prove (i). For each positive integer k, there exists N such that

$$\left|\frac{1}{a_n} - 0\right| < \frac{1}{k}$$

for $n > N$ because $\lim_{n\to\infty} \frac{1}{a_n} = 0$. Then, for $n > N$, $a_n > k$ because $a_n > 0$. This finishes the proof. ☐

EXAMPLE 3.4. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, we have $\lim_{n\to\infty} \sqrt{n} = \infty$. Similarly, since $\lim_{n\to\infty} \sqrt{n} = +\infty$, we have $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

4. Techniques For Computing Limits

THEOREM 4.1. *Let f be a continuous function. Then*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n).$$

IDEA OF PROOF. By the definition of continuity, when $x \to x_0$, $f(x) \to f(x_0)$. Now $\lim\limits_{n\to\infty} a_n = A$ means that $a_n \to A$ when $n \to \infty$. Thus $f(a_n) \to f(A)$ when $n \to \infty$, that is,

$$\lim_{n\to\infty} f(a_n) = f(A) = f(\lim_{n\to\infty} a_n).$$

□

EXAMPLE 4.2.

$$\lim_{n\to\infty} \sin\left(\frac{n\pi}{2n+1}\right) = \lim_{n\to\infty}\left(\frac{\pi}{2+1/n}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

THEOREM 4.3 (L'Hopital's Rule). *Suppose $a_n = f(n)$, $b_n = g(n)$. If $\lim\limits_{n\to\infty} \dfrac{f(n)}{g(n)}$ is of the form $\dfrac{\infty}{\infty}$ or $\dfrac{0}{0}$, then*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

EXAMPLE 4.4. *Show that $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = e^x$.*

$$\lim_{n\to\infty} \ln\left[\left(1+\frac{x}{n}\right)^n\right] = \lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\cdot\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{x}{1+\frac{x}{n}} = x.$$

Thus $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = e^x$.

THEOREM 4.5. *If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist, then*

(1). $\lim\limits_{n\to\infty}(a_n+b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$,
(2). $\lim\limits_{n\to\infty} ka_nk \lim\limits_{n\to\infty} a_n$,
(3). $\lim\limits_{n\to\infty} a_nb_n = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n$,
(4). $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, provided $b_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$.

PROOF. omitted. □

EXAMPLE 4.6. *Find the limit of* $\ln\left(\dfrac{n^2+3n+2}{2+4n+2n^2}\right) + \cos\left(\dfrac{1}{\sqrt{n}}\right)$.

$$\lim_{n\to\infty}\left[\ln\left(\frac{n^2+3n+2}{2+4n+2n^2}\right) + \cos\left(\frac{1}{\sqrt{n}}\right)\right]$$

$$= \lim_{n\to\infty}\left[\ln\left(\frac{(n^2+3n+2)/n^2}{(2+4n+2n^2)/n^2}\right)+\cos\left(\frac{1}{\sqrt{n}}\right)\right]$$
$$= \lim_{n\to\infty}\left[\ln\left(\frac{1+3/n+2/n^2}{2/n^2+4/n+2}\right)+\cos\left(\frac{1}{\sqrt{n}}\right)\right]$$
$$= \ln\left(\frac{1+0+0}{0+0+2}\right)+\cos 0 = \ln\left(\frac{1}{2}\right)+1 = 1-\ln 2.$$

Theorem 4.7 (Some Standard Limits). *Some standard limits are given as follows.*

1. $\lim_{n\to\infty}\frac{1}{n^p}=0$ *for any fixed* $p>0$.
2. $\lim_{n\to\infty}c^n=0$ *for any fixed* c *where* $|c|<1$.
3. $\lim_{n\to\infty}c^{\frac{1}{n}}=1$ *for any fixed* $c>0$.
4. $\lim_{n\to\infty}\sqrt[n]{n}=1$.
5. $\lim_{n\to\infty}\frac{n^p}{c^n}=0$ *for any fixed* p *and* $c>1$.
6. $\lim_{n\to\infty}\frac{c^n}{n!}=0$ *for any fixed* c.
7. $\lim_{n\to\infty}(1+\frac{x}{n})^n=e^x$ *for any fixed* x.
8. $\lim_{n\to\infty}\frac{(\ln n)^p}{n^k}=0$ *for any fixed* $k>0$.

Proof. Assertion 7 was proved in Example 4.4.

1.
$$\lim_{n\to\infty}\frac{1}{n^p}=\left(\lim_{n\to\infty}\frac{1}{n}\right)^p=0^p=0.$$
2. Case 1: When $c=0$, the statement is obvious.
Case 2: When $c>0$, we have
$$\ln\left(\lim_{n\to\infty}c^n\right)=\lim_{n\to\infty}\ln c^n=\lim_{n\to\infty}n\ln c=-\infty.$$
Thus, $\lim_{n\to\infty}c^n=0$.
Case 3: When $c<0$, we have $-|c|^n\le c^n\le |c|^n$ for all n. By Case 2, we have $\lim_{n\to\infty}(-|c|^n)=0=\lim_{n\to\infty}|c|^n$. Hence by Squeeze theorem, we also have $\lim_{n\to\infty}c^n=0$.
3. $\lim_{n\to\infty}c^{\frac{1}{n}}=c^{\lim_{n\to\infty}\frac{1}{n}}=c^0=1$.
4.
$$\ln\left(\lim_{n\to\infty}\sqrt[n]{n}\right)=\lim_{n\to\infty}\ln\sqrt[n]{n}=\lim_{n\to\infty}\frac{\ln n}{n}=0 \text{ (by L'Hopital's rule).}$$
Thus, $\lim_{n\to\infty}\sqrt[n]{n}=e^0=1$.
5. Let k be a fixed positive integer such that $p-k<0$. Then
$$\lim_{n\to\infty}\frac{n^p}{c^n}=\lim_{n\to\infty}\frac{pn^{p-1}}{c^n\ln c}=\lim\frac{p(p-1)n^{p-2}}{c^n(\ln c)^2}=\cdots$$
$$=\lim_{n\to\infty}\frac{p(p-1)\cdots(p-k+1)n^{p-k}}{c^n(\ln c)^k}=\lim_{n\to\infty}\frac{p(p-1)\cdots(p-k+1)}{c^n n^{k-p}(\ln c)^k}=0$$
by L'Hopital's rule.

6. Let $a_n = \dfrac{c^n}{n!} = \dfrac{c \cdot c \cdot \cdots \cdot c}{n(n-1) \cdot \cdots \cdot 1}$. Now fix an integer $M > c$. Then for any $n > M$,

$$0 < a_n = \frac{c \cdot c \cdot \cdots \cdot c}{n(n-1) \cdot \cdots \cdot (M+1)} a_M < \frac{c}{n} a_M.$$

Note that a_M is a fixed number because M is fixed. Since $\lim\limits_{n\to\infty} 0 = 0 = \lim\limits_{n\to\infty} \dfrac{c}{n} a_M$, by the Squeeze theorem, $\lim\limits_{n\to\infty} a_n = 0$.

8. Let $m = \ln n$. Then $n = e^m$. By (5),

$$\lim_{n\to\infty} \frac{(\ln n)^p}{n^k} = \lim_{m\to\infty} \frac{m^p}{e^{km}} = \lim_{m\to\infty} \frac{m^p}{\left(e^k\right)^m} = 0,$$

where $e^k > 1$ because $k > 0$. ☐

Strategy: One can find the limits of many sequences from those of the standard sequences.

Example 4.8. *Find the limits*

1) $\lim\limits_{n\to\infty} \dfrac{8^n + (\ln n)^{10} + n!}{n^6 - n!}$.

2) $\lim\limits_{n\to\infty} \left(1 - \dfrac{1}{2n+1}\right)^{3n}$.

(1).

$$\lim_{n\to\infty} \frac{8^n + (\ln n)^{10} + n!}{n^6 - n!} = \lim_{n\to\infty} \frac{8^n/n! + (\ln n)^{10}/n! + 1}{n^6/n! - 1} = \frac{0+0+1}{0-1} = -1.$$

(2).

$$\lim_{n\to\infty} \left(1 - \frac{1}{2n+1}\right)^{3n} = \lim_{n\to\infty} \left[\left(1 + \frac{-1}{2n+1}\right)^{2n+1}\right]^{\frac{3n}{2n+1}}$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-1}{2n+1}\right)^{2n+1}\right]^{\frac{3}{2+1/n}} = \left(e^{-1}\right)^{\frac{3}{2}} = \frac{1}{e\sqrt{e}}$$

5. Bounded Sets and Sequences

Definition 5.1. A set of real numbers S is *bounded above* if there exists a finite real number M such that

$$x \le M \qquad \forall x \in S.$$

M is called an *upper bound* of S.

Definition 5.2. A set of real numbers S is *bounded below* if there exists a finite real number m such that

$$m \le x \qquad \forall x \in S.$$

m is called a *lower bound* of S.

Definition 5.3. A set which is both bounded above and below is called a *bounded set.*

Remark.
1. Upper bounds and lower bounds are not unique.
2. Some sets only have upper bounds but not lower bounds.
3. Some sets have only lower bounds but not upper bounds.
4. A set which is not bounded is called an *unbounded* set.

Example 5.4. *Let* $S = \{r \mid r \ \text{ is a rational number with } \ r < \sqrt{2}\}$. *Then* S *is bounded above.*

Theorem 5.5. *Every convergent sequence is bounded.*

Proof. Let $\{a_n\}$ be a sequence convergent to A. For $\epsilon = 1$, there exists N such that $|a_n - A| < 1$ or $A - 1 < a_n < A + 1$ for $n > N$. Choose M and m to be the largest and smallest number of the finite numbers

$$a_1, a_2, \ldots, a_N, A+1, A-1,$$

respectively. When $n \le N$, we have $m \le a_n \le M$ because M (m) is the largest (smallest) number of the above finite set. When $n > N$, we have

$$m \le A - 1 < a_n < A + 1 \le M.$$

Thus, for all n, we have $m \le a_n \le M$ and so $\{a_n\}$ is bounded. The proof is finished. ☐

Corollary 5.6 (Test for divergence). *If* $\{a_n\}$ *is unbounded, then* $\{a_n\}$ *diverges.*

Remark.
1. The converse may not be true, i.e., divergent sequence need not be unbounded.
2. The inverse may not be true, i.e., a bounded sequence may not be convergent.

Example. The sequence $\{1, -1, 1, -1, \cdots\}$ is bounded but NOT convergent.

6. Infimum, Supremum

Recall that any finite set of real numbers has a greatest element (maximum) and a least element (minimum).

Example 6.1. $\{-2.5,\ 3.1,\ -4.4,\ 4.5,\ 5\}$

However, this property does not necessarily hold for infinite sets.

Example 6.2. $\{1, 2, 3, 4, \cdots,\}$.

Definition 6.3. A real number M ($\neq \pm\infty$) is called the *least upper bound* or *supremum* of a set E if

(i) M is an upper bound of E, i.e., $x \leq M$ for every $x \in E$, and
(ii) if $M' < M$, then M' is not an upper bound of E (i.e., there is an $x \in E$ such that $M' < x$).

We write $M = \sup E$.

Remark.
(i) $\sup E$ is unique whenever it exists.
(ii) The main difference between $\sup E$ and $\max E$ is that $\sup E$ may not be an element of E, whereas $\max E$ must be an element of E if it does exist).
(iii) If E has a maximum, then $\sup E = \max E$.

Example 6.4. 1. Let $E = \{r \in \mathbb{Q} \mid 0 \leq r \leq \sqrt{2}\}$. Then $\sup E = \sqrt{2}$ but $\max E$ does not exist because $\sqrt{2}$ is not a rational number, that is, $\sup E \notin E$.
2. Let $E = \{1/2, 2/3, 3/4, 4/5, 5/6, \cdots\}$. Then $\sup E = 1$ and $\max E$ does not exist.
3. Let $E = \{1, 1/2, 1/3, 1/4, 1/5, \cdots\}$. Then $\max E = 1 = \sup E$.

Definition 6.5. A real number m ($\neq \pm\infty$) is called the *greatest lower bound* or *infimum* of a set E if

(i) m is a lower bound of E, i.e., $m \leq x$ for every $x \in E$, and
(ii) if $m' > m$, then m' is not a lower bound of E (i.e., there exists an $x \in E$ such that $x < m'$).

We write $m = \inf E$.

Remark.
(i) $\inf E$ is unique whenever it exists.
(ii) The main difference between $\inf E$ and $\min E$ is that $\inf E$ may not be an element of E, whereas $\min E$ must be an element of E if it does exist.
(iii) If E has a minimum, then $\inf E = \min E$.

Example 6.6. 1. Let $E = \{1, 1/2, 1/3, 1/4, \cdots,\}$. Then $\inf E = 0$ but $\min E$ does not exist.
2. Let $E = \{r \in \mathbb{Q} \mid 0 \leq r \leq \sqrt{2}\}$. Then $\min E = \inf E = 0$.

An important property of the set of real numbers is the following

Theorem 6.7 (Completeness Axiom of $\mathbb{R}$). *The following statement hold for subsets of real numbers:*

(i) *If E is bounded above, then* $\sup E$ *exists.*
(ii) *If E is bounded below, then* $\inf E$ *exists.*

Recall that a set E is bounded if and only if it is bounded above and bounded below. Thus the Completeness Axiom leads to

COROLLARY 6.8. *If E is bounded, then both $\sup E$ and $\inf E$ exist.*

7. Monotone Sequences

DEFINITION 7.1. $\{a_n\}$ is called *monotone increasing (decreasing)* if

$$a_n \leq \ (\geq)\ a_{n+1}$$

for every n, that is,

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots$$

$(a_1 \geq a_2 \geq a_3 \geq \cdots)$.

EXAMPLE 7.2. 1. The sequence $\{1/n\}$ is monotone decreasing.
2. The sequence $\{1/2, 2/3, 3/4, 4/5, 5/6, \cdots\}$ is monotone increasing.

PROPOSITION 7.3. *A monotone increasing (decreasing) sequence is bounded below (above).*

PROOF. Let $\{a_n\}$ be a monotone increasing sequence, that is,

$$a_1 \leq a_2 \leq a_3 \leq \cdots .$$

Then a_1 is a lower bound for $\{a_n\}$ and hence the result. □

THEOREM 7.4 (Monotone Convergence Theorem). *Let $\{a_n\}$ be a sequence.*

(i) *If $\{a_n\}$ is monotone increasing and bounded above, then $\{a_n\}$ is convergent and*

$$\lim_{n\to\infty} a_n = \sup_n a_n.$$

(ii) *If $\{a_n\}$ is monotone decreasing and bounded below, then $\{a_n\}$ is convergent and*

$$\lim_{n\to\infty} a_n = \inf_n a_n.$$

PROOF. (i). Suppose $\{a_n\}$ is monotone increasing and bounded above. Then by the Completeness Axiom of $\mathbb{R}$, $\sup\limits_n a_n$ exists (finite). Now, given $\epsilon > 0$, since $\sup\limits_n a_n - \epsilon < \sup\limits_n a_n$, it follows that $\sup\limits_n a_n - \epsilon$ is not an upper bound of $\{a_n\}$. In other words, there exists an integer N such that $a_N > \sup\limits_n a_n - \epsilon$. Then for all $n > N$, we have

$$\sup_n a_n - \epsilon < a_N \leq a_n \leq \sup_n a_n < \sup_n a_n + \epsilon \quad \text{(since } n > N\text{)}.$$

Equivalently, $\left| a_n - \sup\limits_n a_n \right| < \epsilon$ for all $n > N$ and so $\lim\limits_{n\to\infty} a_n = \sup\limits_n a_n$ (exists).

The proof of (ii) is similar. □

EXAMPLE 7.5. Let $a_n = \dfrac{n}{n+1}$, that is, $\{a_n\} = \{1/2, 2/3, 3/4, \cdots\}$. Then a_n is monotone increasing and bounded above. Thus

$$\sup_n a_n = \lim_{n\to\infty} a_n = 1.$$

Corollary 7.6. *If $\{a_n\}$ is monotone increasing (decreasing), then either*

(i) *$\{a_n\}$ is convergent or*
(ii) $\lim_{n\to\infty} a_n = +\infty(-\infty)$.

Proof. Suppose $\{a_n\}$ is monotone increasing, then either $\{a_n\}$ is bounded above or not bounded above.

Case (a): If $\{a_n\}$ is bounded above, then by the Monotone Convergence Theorem, $\{a_n\}$ converges.

Case (b): If $\{a_n\}$ is not bounded above, then $\{a_n\}$ has no upper bounds. Thus for any given $k > 0$, k is not an upper bound of $\{a_n\}$. In other words, there exists N such that

$$a_N > k.$$

Since $\{a_n\}$ is monotone increasing, it follows that for all $n > N$,

$$a_n \geq a_N > k.$$

Therefore, $\lim_{n\to\infty} a_n = +\infty$.

The proof for the case when $\{a_n\}$ is monotone decreasing is similar. □

8. Subsequences

Example 8.1. Find the following subsequences of $\{a_n\} = \{1, -1, 1, -1, 1, -1, \cdots\}$.
$\{a_{2n-1}\} = \{1, 1, 1, \cdots\}$
$\{a_{2n}\} = \{-1, -1, -1, \cdots\}$.

In general, *subsequences* of $\{a_n\}$ are of the form $\{a_{n_k}\}$, $k = 1, 2, 3, ...$, with

$$n_1 < n_2 < n_3 < \cdots .$$

Note. The rule is that we should choose a_{n_1} first and then a_{n_2} with $n_2 > n_1$ and then a_{n_3} with $n_3 > n_2$, so far and so on (up to infinite). Thus n_1 is at least 1, n_2 is at least 2, n_3 is at least 3, $\cdots$.

Theorem 8.2. *Suppose $\lim_{n\to\infty} a_n = A$. Then every subsequence of $\{a_n\}$ also converges to A, ie*

$$\lim_{k\to\infty} a_{n_k} = A.$$

Proof. For any given $\epsilon > 0$, since $\lim_{n\to\infty} a_n = A$, there exists N such that

$$|a_n - A| < \epsilon \quad \text{for all} \quad n > N.$$

Then for all $k > N$, we have

$$n_k \geq k > N.$$

Hence

$$|a_{n_k} - A| < \epsilon \quad \text{for all} \quad k > N.$$

Therefore, $\lim_{k\to\infty} a_{n_k} = A$. □

Theorem 8.3. *Suppose that $\{a_n\}$ has two subsequences that converge to different limits. Then $\{a_n\}$ is divergent.*

Example 8.4. *The sequence $\{1, -1, 1, -1, \cdots\}$ is divergent because $\{a_{2n-1}\} = \{1, 1, \cdots\}$ converges to 1 and $\{a_{2n}\} = \{-1, -1, \cdots\}$ converges to -1.*

9. The $\limsup$ and $\liminf$ of a sequence

Reference: W. Wade, *An introduction to analysis*, Prentice Hall, 1995.

Given a sequence $\{a_n\}$, we can form another sequence $\{b_n\}$ given by

$$b_n = \sup_{k\geq n} a_k = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}.$$

EXAMPLE 9.1. *Let* $\{a_n\} = \{1, -1, 1, -1, \cdots\}$. *Then*

$$b_n = \sup_{k\geq n} a_k = \sup\{\pm 1, \mp 1, \pm 1, \mp 1, \cdots\} = 1.$$

PROPOSITION 9.2. *For any sequence* $\{a_n\}$, *the associated sequence* $\{b_n\} = \{\sup_{k\geq n} a_k\}$ *is always monotone decreasing.*

PROOF. For each n,

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} \geq \sup\{a_{n+1}, a_{n+2}, \cdots\} = b_{n+1}.$$

□

DEFINITION 9.3. The *limit superior* of $\{a_n\}$, denoted by $\limsup a_n$ or $\limsup_{n\to\infty} a_n$ or $\overline{\lim_{n\to\infty}}\, a_n$ is defined to be $\lim_{n\to\infty} b_n$, i.e.

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \sup_{k\geq n} a_k.$$

EXAMPLE 9.4. 1. Let $\{a_n\} = \{1, -1, 1, -1, 1, -1, \cdots\}$.

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} 1 = 1.$$

2. Let $\{a_n\} = \{1, 2, 3, \cdots\}$. Then

$$b_n = \sup_{k\geq n} a_k = \sup\{n, n+1, \cdots\} = +\infty$$

and so $\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = +\infty$.

3. Let $\{a_n\} = \{-1, -2, -3, \cdots\}$. Then

$$b_n = \sup_{k\geq n} a_k = \sup\{-n, -n-1, \cdots\} = -n$$

and so $\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = -\infty$.

PROPOSITION 9.5. *Given any* $\{a_n\}$, $\limsup a_n$ *always exists (either finite,* $+\infty$ *or* $-\infty$*).*

PROOF. If $\{a_n\}$ is not bounded above, then each b_n is $+\infty$, and thus

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = +\infty.$$

If $\{a_n\}$ is bounded above, then each b_n is finite. Since $\{b_n\}$ is monotone decreasing, by Corollary 1.7.3, $\{a_n\}$ converges (to a finite limit), or $\lim_{n\to\infty} b_n = -\infty$. □

Similarly, given any sequence $\{a_n\}$, we can form another sequence $\{c_n\}$ given by

$$c_n = \inf_{k\geq n} a_k = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\}.$$

Definition 9.6. The *limit inferior* of $\{a_n\}$, denoted by $\liminf a_n$ or $\liminf_{n\to\infty} a_n$ or $\underline{\lim}_{n\to\infty} a_n$ is defined to be $\lim_{n\to\infty} c_n$, i.e.

$$\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

Example 9.7. 1. Let $\{a_n\} = \{1, -1, 1, -1, 1, -1, \cdots\}$.

$$\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} (\inf\{\pm 1, \mp 1, \pm 1, \mp 1, \cdots) = \lim_{n\to\infty} -1 = -1.$$

2. Let $\{a_n\} = \{1, 2, 3, \cdots\}$. Then

$$c_n = \inf_{k\geq n} a_k = \inf\{n, n+1, \cdots\} = n$$

and so $\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} c_n = +\infty$.

3. Let $\{a_n\} = \{-1, -2, -3, \cdots\}$. Then

$$c_n = \inf_{k\geq n} a_k = \inf\{-n, -n-1, \cdots\} = -\infty$$

and so $\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} c_n = -\infty$.

Proposition 9.8. *(i). As in Proposition 9.2, for any given sequence* $\{a_n\}$*, the associated sequence* $\{c_n\} = \{\inf_{k\geq n} a_k\}$ *is always monotone increasing.*
(ii). As in Proposition 9.5, for any given $\{a_n\}$*,* $\liminf a_n$ *always exists (either finite,* $+\infty$ *or* $-\infty$*).*

Remark. We always have $\underline{\lim}_{n\to\infty} a_n \leq \overline{\lim}_{n\to\infty} a_n$ because $c_n \leq b_n$.

Proposition 9.9. *(i). If* $\overline{\lim}_{n\to\infty} a_n = B$ *with* $B \neq -\infty$*, then given* $\epsilon > 0$*, there exists* N *such that* $a_n < B + \epsilon$ *for all* $n > N$*.*
(ii). $\underline{\lim}_{n\to\infty} a_n = C$ *with* $C \neq +\infty$*, then given* $\epsilon > 0$*, there exists* N *such that* $a_n > C - \epsilon$ *for all* $n > N$*.*

Proof. (i). If $B = +\infty$, the assertion is obvious and so we assume that B is finite. Since $\overline{\lim}_{n\to\infty} a_n = B$, given any $\epsilon > 0$, there exists N such that for all $n > N$,

$$|b_n - B| < \epsilon \quad \Rightarrow \quad b_n < B + \epsilon \quad \Rightarrow \quad \sup\{a_n, a_{n+1}, \cdots\} < B + \epsilon,$$

i.e. $a_n, a_n + 1, \cdots < B + \epsilon$ for all $n > N$. Proof of (ii) is similar. ☐

Remark. Roughly speaking, Proposition 9.9 says that for any sequence $\{a_n\}$, the a_n's are eventually not much smaller than $\underline{\lim}_{n\to\infty} a_n$ and not much bigger than $\overline{\lim}_{n\to\infty} a_n$.

Proposition 9.10. *If* $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = A$ *(finite), then* $\{a_n\}$ *converges and* $\lim_{n\to\infty} a_n = A$*.*

Proof. Let $b_n = \sup\{a_n, a_{n+1}, \cdots\}$ and let $c_n = \inf\{a_n, a_{n+1}, \cdots\}$. Then

$$c_n = \inf\{a_n, a_{n+1}, \cdots\} \leq a_n \leq b_n = \sup\{a_n, a_{n+1}, \cdots\}.$$

By the assumption, we have

$$\lim_{n\to\infty} c_n = \varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

By the Squeeze theorem, the sequence $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = \varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n.$$

☐

Proposition 9.11. *If $\{a_n\}$ converges, then $\varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.*

Proof. Let $A = \lim_{n\to\infty} a_n$. For any $\epsilon > 0$, there exists N such that $|a_n - A| < \frac{\epsilon}{2}$ or $A - \frac{\epsilon}{2} < a_n < A + \frac{\epsilon}{2}$ for $n > N$. For each (fixed) $n > N$, let $k = n, n+1, n+2, \cdots$, we have

$$A - \frac{\epsilon}{2} \le \inf\{a_n, a_{n+1}, a_{n+2}\cdots\} = c_n \le b_n = \sup\{a_n, a_{n+1}, \cdots\} \le A + \frac{\epsilon}{2}.$$

It follows that

$$|b_n - A| \le \frac{\epsilon}{2} < \epsilon \quad |c_n - A| \le \frac{\epsilon}{2} < \epsilon$$

for all $n > N$. By the $\epsilon - N$ definition,

$$\varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n = A = \lim_{n\to\infty} c_n = \varliminf_{n\to\infty} a_n$$

and hence the result. ☐

10. Cauchy Sequences

Definition 10.1. $\{a_n\}$ is called a *Cauchy sequence* if given any $\epsilon > 0$, there exists a natural number N such that for all m, $n > N$, we have

$$|a_n - a_m| < \epsilon.$$

Remark. Roughly speaking, a sequence is Cauchy if the width of its tail $\to 0$ as $n \to \infty$.

Proposition 10.2. *Every Cauchy sequence is bounded.*

Proof. Let $\{a_n\}$ be a Cauchy sequence. Choose $\epsilon = 1$. There exists N such that $|a_n - a_m| < 1$ for $n, m > N$. In particular, $|a_n - a_{N+1}| < 1$ or

$$a_{N+1} - 1 < a_n < a_{N+1} + 1$$

for $n > N$. Let

$$M = \max\{a_1, a_2, \cdots, a_N, a_{N+1} + 1\}$$
$$m = \min\{a_1, a_2, \cdots, a_N, a_{N+1} - 1\}.$$

For $n \le N$, we have $m \le a_n \le M$, and, for $n > N$, we have

$$m \le a_{N+1} - 1 < a_n < a_{N+1} + 1 \le M.$$

Thus, for all n, we have $m \le a_n \le M$ and so $\{a_n\}$ is bounded. ☐

Theorem 10.3 (Cauchy's criterion). *A sequence is a convergent sequence if and only if it is a Cauchy sequence.*

Proof. $\Rightarrow$, i.e., every convergent sequence is Cauchy.

Given that $\{a_n\}$ is convergent, say $\lim\limits_{n\to\infty} a_n = A$. Then for any given $\epsilon > 0$, there exists N such that

$$|a_n - A| < \frac{\epsilon}{2}$$

for all $n > N$. Now for any $m, n > N$,

$$|a_n - a_m| = |(a_n - A) - (a_m - A)| \le |a_n - A| + |a_m - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

since both $m, n > N$. Therefore, $\{a_n\}$ is a Cauchy sequence.

$\Leftarrow$, i.e., every Cauchy sequence is convergent. Given that $\{a_n\}$ is Cauchy. By Proposition 10.2, $\{a_n\}$ is bounded. As in section 1.9, we let

$$b_n = \sup_{k\ge n} a_k = \sup\{a_n, a_{n+1}, \cdots\}$$

$$c_n = \inf_{k\ge n} a_k = \inf\{a_n, a_{n+1}, \cdots\}.$$

Then $\{b_n\}$ and $\{c_n\}$ are also bounded (check!). Hence both $\overline{\lim\limits_{n\to\infty}}\, a_n = \lim\limits_{n\to\infty} b_n$ and $\underline{\lim\limits_{n\to\infty}}\, a_n = \lim\limits_{n\to\infty} c_n$ exist. Write

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = B \quad \text{and} \quad \underline{\lim_{n\to\infty}}\, a_n = \lim c_n = C.$$

Given any $\epsilon > 0$, since $\{a_n\}$ is a Cauchy sequence, there exists N such that

$$|a_n - a_m| < \frac{\epsilon}{3}$$

for all $m, n > N$. It follows that

$$a_n - \frac{\epsilon}{3} < a_m < a_n + \frac{\epsilon}{3}$$

for all $m, n > N$. In particular, letting $m = n, n+1, n+2, \cdots$, etc., we have

$$a_n - \frac{\epsilon}{3} \le \inf\{a_n, a_{n+1}, a_{n+2} \cdots\} = c_n \le b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} \le a_n + \frac{\epsilon}{3}$$

for $n > N$. Thus

$$-\epsilon < 0 \le b_n - c_n \le \left(a_n + \frac{\epsilon}{3}\right) - \left(a_n - \frac{\epsilon}{3}\right) = \frac{2\epsilon}{3} < \epsilon$$

for $n > N$. By definition, we have $\lim\limits_{n\to\infty}(b_n - c_n) = 0$ and so

$$B - C = \lim_{n\to\infty} b_n - \lim_{n\to\infty} c_n = \lim_{n\to\infty}(b_n - c_n) = 0$$

In other words, we have $\overline{\lim\limits_{n\to\infty}}\, a_n = \underline{\lim\limits_{n\to\infty}}\, a_n$. Then by Proposition 9.10, $\{a_n\}$ converges, and $\lim\limits_{n\to\infty} a_n = B = C$. This finishes the proof. $\square$

CHAPTER 2

Series

1. Series

The expression

$$a_1 + a_2 + a_3 + \cdots$$

written alternatively as $\sum_{k=1}^{\infty} a_k$ is called an *infinite series.*

Example 1.1. (1). $1 + 2 + 3 + 4 + \cdots$.
(2). $1 + 1/2 + 1/3 + 1/4 + \cdots$
(3). $1 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots$.
(4). $1 + 0 + 1 + 0 + 1 + 0 + \cdots$.

Definition 1.2. Given a series $\sum_{k=1}^{\infty} a_k$, its n^{th} *partial sum* S_n is given by

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The sequence $\{S_n\}$ is called the *sequence of partial sums* of the series $\sum_{k=1}^{\infty} a_k$.

Example 1.3. Consider the series $1-1+1-1+1-1+1-1+\cdots$. The $S_{2n-1} = 1$ and $S_{2n} = 0$.

Definition 1.4. Consider the sequence of partial sums $\{S_n\}$ of the series $\sum_{k=1}^{\infty} a_k$. If this sequence converges to a number S, we say that the series $\sum_{k=1}^{\infty} a_k$ converges to S and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S.$$

If $\{S_n\}$ diverges, then we say $\sum_{k=1}^{\infty} a_n$ *diverges.*

Example 1.5 (Geometric Series). Let $a \neq 0$. Consider the series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots .$$

Then the partial sum

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a(1 + r + \cdots + r^{n-1}) = \begin{cases} a\dfrac{1-r^n}{1-r} & r \neq 1 \\ \\ a(n+1) & r = 1 \end{cases}$$

When $-1 < r < 1$, $S_n \to \dfrac{a}{1-r}$ as $n \to \infty$.

When $r > 1$, S_n diverges because $r^n \to +\infty$ as $n \to \infty$.

When $r = 1$, $S_n = a(n+1)$ diverges.

When $r = -1$, $S_n = \dfrac{a[1-(-1)^n]}{2}$ diverges.

When $r < -1$, S_n diverges because $r^n \to \pm\infty$.

Thus the geometric series $\sum\limits_{n=0}^{\infty} ar^n$ converges if and only if $-1 < r < 1$, and,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

for $-1 < r < 1$.

Remark. If $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$ converges, then one always has

(i) $\sum\limits_{k=1}^{\infty}(a_k + b_k) = \sum\limits_{k=1}^{\infty} a_k + \sum\limits_{k=1}^{\infty} b_k$.

(ii) $\sum\limits_{k=1}^{\infty} ca_k = c\sum\limits_{k=1}^{\infty} a_k$.

EXAMPLE 1.6.

$$\begin{aligned}
\sum_{n=1}^{\infty}\left[\left(\frac{1}{4}\right)^n + \left(\frac{1}{5}\right)^n\right] &= \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n \\
&= \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right] + \left[\frac{1}{5} + \left(\frac{1}{5}\right)^2 + \cdots\right] \\
&= \frac{1}{4}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right] + \frac{1}{5}\left[1 + \frac{1}{5} + \left(\frac{1}{5}\right)^2 + \cdots\right] \\
&= \frac{1}{4}\frac{1}{1-\frac{1}{4}} + \frac{1}{5}\frac{1}{1-\frac{1}{5}} \\
&= \frac{1}{4\cdot\frac{3}{4}} + \frac{1}{5\cdot\frac{4}{5}} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.
\end{aligned}$$

THEOREM 1.7. *If* $\sum\limits_{k=1}^{\infty} a_k$ *converges, then* $\lim\limits_{k\to\infty} a_k = 0$.

COROLLARY 1.8 (Divergence Test). *If* $\lim_{n\to\infty} a_n \neq 0$ *(or does not exist), then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

EXAMPLE 1.9. (1). The series $\sum_{n=1}^{\infty}(-1)^n$ is divergent because the limit of the n-th term $(-1)^n$ does not exist.

(2). The series $\sum_{n=1}^{\infty} \frac{n!}{n^2}$ is divergent because

$$\lim_{n\to\infty} \frac{n!}{n^2} = \frac{1}{\lim_{n\to\infty} \frac{n^2}{n!}} = \frac{1}{0} = +\infty \neq 0.$$

(3). The series $\sum_{n=1}^{\infty} \frac{2n+1}{3n+2}$ is divergent because

$$\lim_{n\to\infty} \frac{2n+1}{3n+2} = \lim_{n\to\infty} \frac{2+1/n}{3+2/n} = \frac{2}{3} \neq 0.$$

Remark. The divergence test is a "one-way" test, i.e., $\lim_{n\to\infty} a_n = 0$ does NOT imply $\sum_{n=1}^{\infty} a_n$ converges.

EXAMPLE 1.10 (Harmonic Series). $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$. But we show that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Now $S_n = \sum_{k=1}^{n} \frac{1}{k}$.

$$\begin{aligned} S_{2^n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^{n-1}+1}+\frac{1}{2^{n-1}+2}+\cdots+\frac{1}{2^n}\right) \\ &\geq 1+\frac{1}{2}+2\cdot\frac{1}{4}+4\cdot\frac{1}{8}+8\cdot\frac{1}{16}+\cdots 2^{n-1}\cdot\frac{1}{2^n} \\ &= 1+n\cdot\frac{1}{2}. \end{aligned}$$

Thus, the subsequence $\{S_{2^n}\}$ is unbounded. So $\{S_n\}$ is also unbounded. Hence, $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent

2. Tests for Positive Series

Goal: Given a series, we want to be able to test if it converges or not.

2.1. Positive Series.

Definition 2.1. A series $\sum_{k=1}^{\infty} a_k$ is called a *positive series* if every term a_k is positive.

We first develop tests for positive series.

Proposition 2.2. *For a positive series* $\sum_{k=1}^{\infty} a_k$*, the sequence of partial sums* $\{S_n\}$ *is monotone increasing.*

Proof. This is so since $S_{n+1} - S_n = a_{n+1} > 0$ for each n. ☐

Corollary 2.3. *(i) If* $\{S_n\}$ *is bounded for a positive series, then the series converges.*
(ii) If $\{S_n\}$ *is not bounded from above, then the series diverges.*

2.2. Comparison Test.

Theorem 2.4 (Comparison Test). *Consider 2 positive series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$*. Suppose that eventually* $0 \leq a_k \leq b_k$*.*

(i) *If* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*
(ii) *If* $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Proof. Assertion (ii) follows immediately from (i). Let $A_n = \sum_{k=1}^{n} a_k$, and $B_n = \sum_{k=1}^{n} b_k$. Then $A_n \leq B_n$ for all n. Suppose that $\sum_{k=1}^{\infty} b_k$ converges, that is, $\sum_{k=1}^{\infty} b_k$ is a (finite) number. Then

$$A_n \leq B_n \leq \sum_{k=1}^{\infty} b_k$$

for all n and so A_n is bounded above. By Corollary 2.3, $\sum_{k=1}^{\infty} a_k$ is convergent. ☐

Example 2.5. (1). The series $\sum_{k=1}^{\infty} \left(\frac{2k-1}{3k+2}\right)^k$ converges because

$$\left(\frac{2k-1}{3k+2}\right)^k \leq \left(\frac{2}{3}\right)^k$$

and the geometric series $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ converges.

(2). The series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges because

$$\frac{1}{\sqrt{k}} \geq \frac{1}{k}$$

and the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Remark. 1. Suppose $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges. Then NO conclusion can be drawn.
2. Similarly, suppose $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} a_n$ converges. Then NO conclusion can be drawn.

Example 2.6.

$$0 \leq \left(\frac{1}{2}\right)^n \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. Now $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Corollary 2.7 (Limit Comparison Test). *If* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are (eventually) positive, and*

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L(\neq 0, \neq \infty),$$

then either both series converge or both series diverge.

Proof. Let $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$. Since

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L\ (\neq 0, \neq \infty),$$

$\left\{\frac{a_k}{b_k}\right\}$ is bounded above, say, by M. Thus

$$0 \leq a_k \leq M b_k$$

for all k. Similarly, since $\lim_{k\to\infty} \frac{b_k}{a_k} = \frac{1}{L}\ (\neq 0, \neq \infty)$, $\left\{\frac{b_k}{a_k}\right\}$ is bounded above, say, by M'. Thus $0 \leq b_k \leq M' a_k$ for all k.

If $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} M' a_k = M' \sum_{k=1}^{\infty} a_k$ is also convergent. By the comparison test, it follows that $\sum_{k=1}^{\infty} b_k$ is also convergent because $b_k \leq M' a_k$.

If $\sum_{k=1}^{\infty} b_k$ is convergent, then $\sum_{k=1}^{\infty} Mb_k = M\sum_{k=1}^{\infty} b_k$ is also convergent. By the comparison test, it follows that $\sum_{k=1}^{\infty} a_k$ is also convergent because $a_k \leq Mb_k$. Hence $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=1}^{\infty} b_k$ is convergent. Hence $\sum_{k=1}^{\infty} a_k$ is divergent if and only if $\sum_{k=1}^{\infty} b_k$ is divergent. □

Standard series used in comparison and limit comparison tests.

1. The Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{converges} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

2. The p-series: for a fixed p,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

To be proved in the section on Integral Test.

Example 2.8. *Determine the convergence or divergence:*

1) $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

2) $\sum_{n=1}^{\infty} \frac{n^3+\ln n+8}{n^4-2n+3}$

(1). It is convergent, by the comparison test, because

$$0 \leq \frac{1+\cos n}{n^2} \leq \frac{2}{n^2}$$

and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(2). It is divergent, by the limit comparison test, because

$$\lim_{n\to\infty} \frac{\dfrac{n^3+\ln n+8}{n^4-2n+3}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^4+n\ln n+8n}{n^4-2n+3} = \lim_{n\to\infty} \frac{1+\dfrac{\ln n}{n^3}+\dfrac{1}{n^3}}{1-\dfrac{2}{n^3}+\dfrac{3}{n^4}} = \frac{1+0+0}{1-0+0} = 1$$

and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

COROLLARY 2.9. *[Limit Comparison Test for "$a_n << b_n$"] Suppose* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are eventually positive, and* $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$.

(i) *If* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_n$ *converges.*

(ii) *If* $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_n$ *diverges.*

PROOF. $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ implies for every $\epsilon > 0$, there is an N such that

$$\left|\frac{a_k}{b_k} - 0\right| < \epsilon \quad \forall k > N.$$

We choose $\epsilon = 1$. Then the above inequality is

$$a_k < b_k \quad \forall k > N.$$

We get the result by applying the comparison test. □

Remark. If $\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$, interchange a_n and b_n, and then apply Corollary 2.9.

EXAMPLE 2.10. *Determine convergence or divergence of* $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^k}$, *where* k *is a constant.*Let $b_n = \frac{1}{(\ln n)^k}$ and let $a_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{(\ln n)^k}} = \lim_{n\to\infty} \frac{(\ln n)^k}{n} = 0.$$

Since the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^k}$ is divergent for any k.

2.3. Integral Test.

THEOREM 2.11 (Integral Test). *Let* $f\colon [1,\infty) \to \mathbb{R}$ *be a positive, continuous and monotone decreasing function. Suppose we have a series* $\sum_{k=1}^{\infty} a_k$ *such that* $a_k = f(k)$, *then the series* $\sum_{k=1}^{\infty} a_k$ *and the integral* $\int_1^{\infty} f(x)dx$ *either both converge or both diverge.*

Review on improper integrals

Let $f(x) : [1, \infty) \to [0, \infty)$ be a continuous function. Then one defines the improper integral

$$\int_1^\infty f(x)\, dx = \lim_{n\to\infty} \int_1^n f(x)\, dx = \text{area under } f(x) \text{ over } [1, \infty).$$

Here we say that $\displaystyle\int_1^\infty f(x)\, dx$ converges if the limit $\displaystyle\lim_{n\to\infty} \int_1^n f(x)\, dx$ exists (finite), i.e., the area under $f(x)$ over $[1, \infty)$ is finite.

We also say that $\displaystyle\int_1^\infty f(x)\, dx$ diverges if the limit $\displaystyle\lim_{n\to\infty} \int_1^n f(x)\, dx$ does not exist.

The improper integrals can be defined similarly for integrals over other intervals such as $[0, \infty)$, $(-\infty, 0]$.

Proof of the Integral test. Let $f(x)$ be a positive, continuous, monotone decreasing function such that $a_n = f(n)$ for all n.

From the graph, we see that

$$\text{area of the rectangles } \le \text{area under } f(x) \text{ over } [1, n], \text{ i.e.,}$$

$$\sum_{k=2}^{n} f(k) \le \int_1^n f(x)dx \le \int_1^\infty f(x)dx.$$

Thus, if $\displaystyle\int_1^\infty f(x)\, dx < \infty$, then

$$\sum_{k=2}^{n} a_k = \sum_{k=2}^{n} f(k) \le \int_1^\infty f(x)\, dx < \infty,$$

i.e., for all n, $\displaystyle\sum_{k=2}^{n} a_k$ is bounded above by the finite number $\displaystyle\int_1^\infty f(x)\, dx$. Since we also have $a_k \ge 0$, it follows from Corollary 2.3 that $\displaystyle\sum_{k=2}^{\infty} a_k$ converges, and thus $\displaystyle\sum_{k=1}^{\infty} a_k$ also converges. Next we consider the following graph:

From the graph, it is easy to see that

area under the rectangles $\geq$ area under $f(x)$ over $[1, n]$, i.e., $\displaystyle\sum_{k=1}^{n-1} f(k) \geq \int_1^n f(x)\, dx$.

Thus, if $\displaystyle\sum_{k=1}^{\infty} a_k < \infty$, then

$$\infty > \sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{n-1} a_k = \sum_{k=1}^{n-1} f(k) \geq \int_1^n f(x)\, dx,$$

i.e., $\displaystyle\int_1^n f(x)\, dx$ is bounded above by the finite number $\displaystyle\sum_{k=1}^{\infty} a_k < \infty$. Letting $n \to \infty$, it follows that we have

$$\int_1^{\infty} f(x)\, dx \leq \sum_{k=1}^{\infty} a_k < \infty.$$

In conclusion, we have $\displaystyle\sum_{k=1}^{\infty} a_k$ converges if and only if $\displaystyle\int_1^{\infty} f(x)\, dx$ converges, which also means that $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges if and only if $\displaystyle\int_1^{\infty} f(x)\, dx$ diverges. ☐

EXAMPLE 2.12. *Show that*

1) *the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$.

2) *the series* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^k}$ *converges if and only if* $k > 1$.

(1). If $p \leq 0$, then $\dfrac{1}{n^p}$ does not tend to 0 and so, by divergence test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Assume that $p > 0$. Let $f(x) = \dfrac{1}{x^p}$ on $[1, +\infty)$. Then $f(x)$ is continuous, positive and monotone decreasing. Now

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} x^{-p}\, dx = \begin{cases} \left.\dfrac{1}{-p+1} x^{-p+1}\right|_1^{+\infty} & p \neq 1 \\ \\ \ln(+\infty) - \ln 1 & p = 1 \end{cases}$$

Thus $\displaystyle\int_1^{\infty} f(x)\, dx$ converges if and only if $p > 1$ and so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

(2). Let $f(x) = \dfrac{1}{x(\ln x)^k}$ on $[2, +\infty)$. Then $f(x)$ is continuous, positive. We check that $f(x)$ is *eventually* monotone deceasing. From

$$f'(x) = \left(x^{-1}(\ln x)^{-k}\right)' = -x^{-2}(\ln x)^{-k} - kx^{-1}(\ln x)^{-k-1}\frac{1}{x} = -x^{-2}(\ln x)^{-k-1}(\ln x + k),$$

we have $f'(x) \leq 0$ when $\ln x > -k$. Thus $f(x)$ is monotone decreasing when $\ln x > -k$ and so $f(x)$ is eventually monotone decreasing. Now

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x(\ln x)^k}\,dx \,\underset{dy = \frac{1}{x}dx}{\overset{y=\ln x}{=\!=\!=\!=}}\, \int_{\ln 2}^\infty \frac{1}{y^k}dy = \begin{cases} \left.\dfrac{1}{-k+1}y^{-k}\right|_{\ln 2}^{\infty} & k \neq 1 \\ \ln(+\infty) - \ln(\ln 2) & k = 1 \end{cases}$$

Thus $\displaystyle\int_2^\infty f(x)\,dx$ converges if and only if $k > 1$ and so the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^k}$ converges if and only if $k > 1$.

2.4. Ratio Test.

THEOREM 2.13 (Ratio Test). *Consider the positive series* $\displaystyle\sum_{n=1}^\infty a_n$. *Suppose*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell. \tag{1}$$

(i) *If* $0 \leq \ell < 1$, *then* $\displaystyle\sum_{n=1}^\infty a_n$ *converges.*
(ii) *If* $1 < \ell \leq \infty$, *then* $\displaystyle\sum_{n=1}^\infty a_n$ *diverges.*
(iii) *If* $\ell = 1$, *then the test is inconclusive.*

PROOF. We will prove (i) and (ii). Given any $\epsilon > 0$, it follows from (1) that there exists N such that for all $n > N$,

$$\left|\frac{a_{n+1}}{a_n} - \ell\right| < \epsilon \quad \text{or} \quad \ell - \epsilon < \frac{a_{n+1}}{a_n} < \ell + \epsilon.$$

By repeating using the above inequalities, it follows that for all $m > 0$,

$$a_{N+1}(\ell-\epsilon)^m < a_{N+1+m} < a_{N+1}(\ell+\epsilon)^m. \tag{2}$$

(i). If $\ell < 1$, choose $\epsilon > 0$ such that $\ell + \epsilon < 1$, then $\displaystyle\sum_{m=1}^\infty a_{N+1}(\ell+\epsilon)^m$ converges (since it is a geometric series with common ratio satisfying $|r| = \ell + \epsilon < 1$). Together with the right-hand-side of (2), it follows from the comparison test that $\displaystyle\sum_{m=1}^\infty a_{N+1+m}$ converges, and thus $\displaystyle\sum_{n=1}^\infty a_n$ converges.

(ii). If $\ell > 1$, choose $\epsilon > 0$ such that $\ell - \epsilon > 1$, then by the left-hand-side of (2), we have, for all $m > 0$,

$$a_{N+1+m} \geq a_{N+1}(\ell-\epsilon)^m > a_{N+1} > 0.$$

In particular, $\displaystyle\lim_{n\to\infty} a_n \neq 0$ or does not exist. By the divergence test, $\displaystyle\sum_{n=1}^\infty a_n$ diverges.

$\square$

EXAMPLE 2.14. *Determine convergence or divergence.*

1) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

2) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

(1).

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{n! \cdot \dfrac{(n+1)^n}{n^n} \cdot (n+1)} = \lim_{n\to\infty} \frac{1}{\left(1+\dfrac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

Thus the series converges.

(2).

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{[(n+1)!]^2}{(2n+2)!}}{\dfrac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2/n^2}{(2n+2)(2n+1)/n^2} = \lim_{n\to\infty} \frac{1+2/n+1/n^2}{(2+2/n)(2+1/n)} = \frac{1+0+0}{(2+0)\cdot(2+0)} = \frac{1}{4} < 1.$$

Thus the series converges.

2.5. Root Test.

THEOREM 2.15. *Consider the series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *with each* $a_n \geq 0$, *and let*

$$\ell = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}. \tag{3}$$

(i) *If* $0 \leq \ell < 1$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) *If* $1 < \ell \leq \infty$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges.*

(iii) *If* $\ell = 1$, *then the test is inconclusive.*

PROOF. We will prove (i) and (ii).

(i) Suppose that $\ell < 1$. Then for all given $\epsilon > 0$, it follows from (3) and Proposition 9.9 of chapter 1, that there exists an N such that $\sqrt[n]{a_n} < \ell + \epsilon$ for all $n > N$. Now choose $\epsilon > 0$ s.t. $\ell + \epsilon < 1$. Then

$$0 \leq a_n < (\ell+\epsilon)^n \quad \text{for all } n > N. \tag{4}$$

Since $\displaystyle\sum_{n=1}^{\infty} (\ell+\epsilon)^n$ converges (as it is a geometric series with common ratio satisfying $|r| = \ell + \epsilon < 1$), it follows from (4) and the comparison test that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(ii). We are going to prove (ii) by contradiction. Given that $\ell > 1$. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then by the divergence test, we have $\lim_{n\to\infty} a_n = 0$. In particular, there exists N such that $0 \le a_n < 1$ for all $n > N$. Hence $\sqrt[n]{a_n} < 1$ for all $n > N$, and it follows that we must have $\ell \le 1$, which is a contradiction. Hence $\sum_{n=1}^{\infty} a_n$ diverges. ☐

Corollary 2.16 (Simplified Root Test). *Consider the series* $\sum_{n=1}^{\infty} a_n$ *with each* $a_n \ge 0$. *Suppose that* $\lim_{n\to\infty} \sqrt[n]{a_n} = \ell$.

(i) *If* $0 \le \ell < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*
(ii) *If* $1 < \ell \le \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
(iii) *If* $\ell = 1$, *then the test is inconclusive.*

Proof. We will prove (i) and (ii). Recall from Proposition 9.11 of chapter 1 that if $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}$. Then the Corollary follows from Theorem 2.15. ☐

Example 2.17. *Determine convergence or divergence of the series*

$$\sum_{n=1}^{\infty} 2^n \left(1 - \frac{1}{n}\right)^{n^2}.$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[2^n \left(1 - \frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}} = \lim_{n\to\infty} 2\left(1 - \frac{1}{n}\right)^{n} = \frac{2}{e} < 1.$$

Thus the series converges.

Example 2.18. *Determine convergence or divergence of the series*

$$\sum_{n=1}^{\infty} (3 + \sin n)^n \left(1 - \frac{2}{n}\right)^{n^2}.$$

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = \overline{\lim_{n\to\infty}} \left[(3 + \sin n)^n \left(1 - \frac{2}{n}\right)^{n^2}\right]^{\frac{1}{n}}$$

$$= \overline{\lim_{n\to\infty}} (3 + \sin n) \left(1 - \frac{2}{n}\right)^{n} \le \overline{\lim_{n\to\infty}}\, 4\left(1 - \frac{2}{n}\right)^{n} = \frac{4}{e^2} < 1.$$

Thus the series converges.

3. Series with both +ve and -ve terms

Example 3.1. Consider the following series.

1. $2+4-6-8+10+12-14-16+\cdots$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2}=-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\cdots$

Definition 3.2. An *alternating series* is of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+\cdots, \quad \text{or}$$

$$\sum_{n=1}^{\infty}(-1)^{n}a_n=-a_1+a_2-a_3+a_4-\cdots$$

with each $a_n>0$.

Example 3.3.

$$1-1+1-1+1-1+1-1+\cdots$$
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$
$$-1+2-3+4-5+6-\cdots$$

Theorem 3.4 (The Alternating Series test). *Let* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *(or* $\sum_{n=1}^{\infty}(-1)^{n}a_n$*) be an alternating series. Suppose that*

(i) $a_n>0$ *for all* n,
(ii) a_n *is monotone decreasing (i.e.,* $a_n\ge a_{n+1}$ *for all* n*), and*
(iii) $\lim_{n\to\infty}a_n=0$.

Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *(resp.* $\sum_{n=1}^{\infty}(-1)^{n}a_n$*) is convergent.*

Proof. We look at the partial sums $\{S_n\}$ of the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. Now,

$$S_{2n}=a_1-a_2+a_3-a_4+\cdots+a_{2n-1}-a_{2n}$$
$$\le a_1-a_2+a_3-a_4+\cdots+a_{2n-1}-a_{2n}+a_{2n+1}-a_{2n+2}=S_{2n+2}.$$

Thus $\{S_{2n}\}$ is monotone increasing. Also,

$$S_{2n}=a_1-a_2+a_3-a_4+a_5-a_6\cdots+a_{2n-1}-a_{2n}\le a_1.$$

Thus $\{S_{2n}\}$ is bounded above by a_1. Then by Monotone Convergence theorem, $\{S_{2n}\}$ is convergent, and write $\lim_{n\to\infty}S_{2n}=S$. For any $\epsilon>0$, there exists N_1 and N_2 such that $|S_{2n}-S|<\frac{\epsilon}{2}$ for $n>N_1$ and $|a_n-0|<\frac{\epsilon}{2}$ for $n>N_2$. Choose $N=\max\{2N_1+1,N_2\}$. We show that $|S_n-S|<\epsilon$ for all $n>N$.

Case I. $n = 2k$ with $n > N$. Then *displaystylek* $> \frac{N}{2} > N_1$. Then $|S_{2k} - S| < \frac{\epsilon}{2} < \epsilon$ by the construction of N_1.

Case II. $n = 2k+1$ with $n > N$. Then $k > \dfrac{N-1}{2} \geq N_1$ and $n = 2k+1 > N \geq N_2$. Thus

$$|S_{2k} - S| < \frac{\epsilon}{2} \quad \text{and} \quad |a_{2k+1}| < \frac{\epsilon}{2}.$$

Now

$$|S_n - S| = |S_{2k+1} - S| = |S_{2k} + a_{2k+1} - S| \leq |S_{2k} - S| + |a_{2k+1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $|S_n - S| < \epsilon$ for all $n > N$ and so $\lim\limits_{n\to\infty} S_n = S$ (convergent), and thus $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent. ☐

Example 3.5. *Show that convergence or divergence of the series*

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p} = \begin{cases} \textit{convergence} & p > 0 \\ \\ \textit{divergence} & p \leq 0. \end{cases}$$

Proof. If $p \leq 0$, then the n-th term $(-1)^n\dfrac{1}{n^p}$ does not tend to 0. Thus the series diverges in this case by the divergence test.

Assume that $p > 0$. Let $a_n = \dfrac{1}{n^p}$. Then $a_n > 0$, monotone decreasing and $\lim\limits_{n\to\infty} a_n = 0$. Thus the alternating series converges in this case.

In conclusion, we have that the series $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{1}{n^p}$ converges when $p > 0$ and diverges when $p \leq 0$. ☐

4. Absolute and Conditional Convergence

Theorem 4.1. *If* $\sum\limits_{n=1}^{\infty}|a_n|$ *converges, then* $\sum\limits_{n=1}^{\infty}a_n$ *converges.*

Proof. Let $T_n = \sum\limits_{k=1}^{n}|a_k|$, $S_n = \sum\limits_{k=1}^{n}a_k$. Since $\{T_n\}$ converges, $\{T_n\}$ is Cauchy. Thus, for any $\epsilon > 0$, there is a N such that $|T_n - T_m| < \epsilon$ for all $n, m > N$. For any $n, m > N$, we may assume that $m \geq n$, say $m = n + p$ (as one of them should be greater than another). Then

$$|S_n - S_m| = |S_n - (S_n + a_{n+1} + a_{n+2} + \cdots + a_{n+p})| = |a_{n+1} + a_{n+2} + \cdots + a_{n+p}|$$
$$\leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| = T_m - T_n = |T_n - T_m| < \epsilon.$$

Thus $\{S_n\}$ is a Cauchy sequence and so it converges. Thus the series $\sum\limits_{n=1}^{\infty}a_n$ converges and hence the result. ☐

EXAMPLE 4.2. *Determine convergence or divergence of the series*

$$\sum_{n=2}^{\infty} \frac{\sin n}{n(\ln n)^2}.$$

Since

$$\left| \frac{\sin n}{n(\ln n)^2} \right| \le \frac{1}{n(\ln n)^2}$$

and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by Example 2.12, the series $\displaystyle\sum_{n=2}^{\infty} \left| \frac{\sin n}{n(\ln n)^2} \right|$ converges. Thus the series $\displaystyle\sum_{n=2}^{\infty} \frac{\sin n}{n(\ln n)^2}$ converges.

DEFINITION 4.3. We say that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges.

Remark. If you are testing for absolute convergence, all the techniques for the positive series are applicable.

COROLLARY 4.4. *Every absolutely convergent series is convergent.*

PROOF. $\displaystyle\sum_{n=1}^{\infty} a_n$ absolutely convergent implies $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges. By Theorem 4.1, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. ☐

Q: Is the converse of the Corollary true? I.e., if a series is convergent, will it be absolutely convergent?
A: No, it is not necessarily true.

EXAMPLE 4.5. The series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges by Example 3.5, but it is NOT absolutely convergent by the p-series.

DEFINITION 4.6. $\displaystyle\sum_{n=1}^{\infty} a_n$ is said to be *conditionally convergent* if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges but $\displaystyle\sum_{n=1}^{\infty} |a_n|$ diverges.

EXAMPLE 4.7. The series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ is conditionally convergent.

THEOREM 4.8. *Every series is either absolutely convergent, conditionally convergent or divergent.* ☐

Example 4.9. *The series*

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}=\begin{cases}\textit{absolutely convergence} & p>1\\ \textit{conditionally convergence} & 0<p<1\\ \textit{divergence} & p\leq 0\end{cases}$$

5. Remarks on the various tests for convergence/divergence of series

1. **n-th term test for divergence:**

- a test for divergence ONLY, and it works for series with positive and negative terms, e.g. $\sum_{n=1}^{\infty}(-1)^n$.

2. **Comparison test/Limit Comparison test:**

- when applying these tests, one usually compares the given series with a geometric series or a p-series.
- generally works for series which look like the geometric series or the p-series, e.g. $\sum_{n=1}^{\infty}\frac{2+(-1)^n}{4^n}$, $\sum_{n=1}^{\infty}\frac{2^{\frac{1}{n}}}{n^2}$.
- when an oscillating factor/term appears, e.g. $\sum_{n=1}^{\infty}\frac{2+(-1)^n}{3^n}$, try the Comparison test rather than the Limit Comparison test.

3. **Integral test:**

e.g. $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$.

4. **Ratio test:**

- generally works for series which look like the geometric series, series with $n!$, and certain series defined recursively, e.g. $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$, $\sum_{n=1}^{\infty}\frac{(2n)!}{4^n\cdot n!}$, $\sum_{n=1}^{\infty}a_n$, where $a_1=1$, $a_n=(\frac{1}{2}+\frac{1}{n})a_{n-1}$, $n=2,3,\cdots$.

5. **(Simplified) Root test:**

- generally works for series where a_n involves a high power such as the n-th power, e.g. $\sum_{n=1}^{\infty}\frac{n}{3^n}$, $\sum_{n=1}^{\infty}2^n\Big(1-\frac{1}{n}\Big)^{n^2}$.

6. **Alternating Series test:** - works for alternating series only, e.g. $\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}$.

Remark. In general, Tests 2 - 5 works only for $\sum_{n=1}^{\infty}a_n$, where $a_n\geq 0$.

6. Estimation of Series

Let $\sum_{n=1}^{\infty} a_n$ be a series of numbers. Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent. We are going to estimate the infinite sum $S = \sum_{n=1}^{\infty} a_n$.

Example. Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$. By taking the partial sum, we have

$$S \approx S_1 = 1,\ S \approx S_2 = 1 + \frac{1}{2^2} = 1.25,\ S \approx S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2} \approx 1.361, \ldots.$$

By using computer programs, we are able to compute much more, say $S_{1000000}$. A mathematical problem is then what is the 'error' for estimating S by using the partial sum S_n. In other words, how to estimate the remainder

$$R_n = |S - S_n| = |a_{n+1} + a_{n+2} + \cdots|.$$

Theorem [Integral Test Estimation]. *Let $f : [1, \infty) \to \mathbb{R}$ be a positive, continuous and monotone decreasing function. Suppose we have a series $\sum_{k=1}^{\infty} a_k$ such that $a_k = f(k)$, then the remainder*

$$R_n = |a_{n+1} + a_{n+2} + \cdots| \leq \int_n^{\infty} f(x)dx.$$

Proof. From the graph,

we see that

area of the rectangles $\leq$ area under $f(x)$ over $[n, \infty)$, that is,

$$R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)dx.$$

Example 1. $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1 + \frac{1}{2^2} + \ldots + \frac{1}{10^2}$ with the error

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2} dx = \frac{1}{10} = 0.1.$$

To get the estimation with error less than or equal to 0.01, we may need to add up the first 100 terms.

Example 2.*Estimate* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$ *with error* ≤ 0.001.

From $\displaystyle\int_n^{\infty}\frac{1}{x^4}dx \leq 10^{-3}$, we have $\frac{1}{3}n^{-3} \leq 10^{-3}$ or $n^3 \geq \frac{10^3}{3}$ or $n \geq 7$. Thus

$$\sum_{n=1}^{\infty}\frac{1}{n^4} \approx 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4}$$

with error ≤ 0.001.

Theorem [Alternating Series Estimation]. *Let* $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *(or* $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$*) be an alternating series. Suppose that*

(i) $a_n > 0$ *for all* n,
(ii) a_n *is monotone decreasing* *(i.e.,* $a_n \geq a_{n+1}$ *for all* n*), and*
(iii) $\lim_{n\to 0} a_n = 0$.

Then the remainder

$$R_n = \left|\sum_{k=n+1}^{\infty}(-1)^{k+1}a_k\right| \leq a_{n+1}$$

(resp. $R_n = \left|\displaystyle\sum_{k=n+1}^{\infty}(-1)^k a_k\right| \leq a_{n+1}$*).*

Proof.

$$\begin{aligned} R_n &= \left|(-1)^{n+2}a_{n+1}+(-1)^{n+3}a_{n+2}+\cdots\right| \\ &= \left|a_{n+1}-a_{n+2}+a_{n+3}-a_{n+4}+\cdots\right|. \end{aligned}$$

Since

$$\begin{aligned} &a_{n+1}-a_{n+2}+a_{n+3}-a_{n+4}+\cdots \\ &= a_{n+1}-(a_{n+2}-a_{n+3})-(a_{n+4}-a_{n+5})-(a_{n+6}-a_{n+7})-\cdots \leq a_{n+1} \end{aligned}$$

and

$$a_{n+1}-a_{n+2}+a_{n+3}-a_{n+4}+\cdots = (a_{n+1}-a_{n+2})+(a_{n+3}-a_{n+4})+\cdots \geq 0,$$

we have $R_n \leq a_{n+1}$.

Example 3. *Estimate* $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ *with error within* 0.001.

From $\dfrac{1}{(n+1)^4} \leq 10^{-3}$, we have $n+1 \geq 6$ or $n \geq 5$. Thus

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4} \approx 1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}$$

with error within 0.001.

CHAPTER 3

Series of Functions

1. Sequence of Functions

Let I be an interval in $\mathbb{R}$, e.g. $(-1,1), [0,1]$, etc. For each $n \in \mathbb{N}$, let $F_n : I \to \mathbb{R}$ be a function. Then we say $\{F_n\}$ forms a *sequence of functions* on I.

Example 1.1. 1. $F_n(x) = x^n, \qquad 0 < x < 1$. Then $\{F_n\}$ forms a sequence of functions on (0,1).
2. $\left\{\left(1+\dfrac{x}{n}\right)^n\right\}$ forms a sequence of functions on $(-\infty, \infty)$.

Write out some terms:

$$F_1(x) = 1 + x \quad F_2(x) = \left(1+\frac{x}{2}\right)^2 \quad F_3(x) = \left(1+\frac{x}{3}\right)^3$$

If we fix the x, and let $n \to \infty$,

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

So for each x, we can define

$$F(x) = \lim_{n\to\infty} F_n(x).$$

Definition 1.2. A sequence $\{F_n\}$ is said to *converge pointwise* to a function F on I if

$$\lim_{n\to\infty} F_n(x) = F(x) \qquad \text{for each } x \in I,$$

i.e., for each $x \in I$ and given any $\epsilon > 0$, there exists an N (which depends on x and ϵ) such that

$$|F_n(x) - F(x)| < \epsilon \qquad \forall n > N.$$

Definition 1.3. $\{F_n\}$ is said to *converge pointwise on I* if $\{F_n\}$ converge pointwise to some function F.

Remark. The function F is called the *limiting function* of $\{F_n\}$ and is necessarily unique.

Example 1.4. *Does the sequence $\{x^n\}$ where $0 < x < 1$ have a limiting function?*

$$F(x) = \lim_{n\to\infty} x^n = 0$$

for $0 < x < 1$.

Remarks on pointwise convergence. Suppose a sequence of functions $\{F_n\}$ converges pointwise to a function F on the interval $[a, b]$. We want to know whether the limiting function F inherits the properties of $\{F_n\}$. For example, we may ask the following questions:

Question 1: Suppose each F_n is a continuous function on $[a, b]$. Is it true that F is necessarily continuous on $[a, b]$?
Question 2: Is it true that

$$\lim_{n\to\infty} \int_a^b F_n(x)\,dx = \int_a^b F(x)\,dx?,$$

i.e. whether

$$\lim_{n\to\infty} \int_a^b F_n(x)\,dx = \int_a^b \Big(\lim_{n\to\infty} F_n(x)\Big)\,dx?$$

Answers: NO to both questions.

Example 1.5 (Counter-example to Question 1). Consider the functions

$$F_n(x) = x^n, \quad x \in [0, 1].$$

For each fixed $x \in [0, 1)$, we have

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} x^n = 0 \quad (\text{since } |x| < 1).$$

At $x = 1$, we have

$$\lim_{n\to\infty} F_n(1) = \lim_{n\to\infty} 1^n = 1.$$

Thus $\{F_n\}$ converges pointwise to the function F on the interval $[0, 1]$ given by

$$F(x) = \begin{cases} 0, & \text{for } x \in [0, 1), \\ \\ 1, & \quad x = 1. \end{cases}$$

Each F_n is continuous on the whole interval $[0, 1]$, but F is not continuous at $x = 1$.

Example 1.6 (Counter-example to Question 2). Consider the functions

$$F_n(x) = \begin{cases} n^2 x, & 0 < x < \frac{1}{n}, \\ \\ 2n - n^2 x, & \frac{1}{n} \le x < \frac{2}{n}, \\ \\ 0, & \frac{2}{n} \le x < 1. \end{cases}$$

For each fixed $x \in (0, 1]$, one sees that $F_n(x) = 0$ whenever $n \ge \frac{2}{x}$, and hence

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} 0 = 0.$$

Also, at $x = 0$, we have

$$\lim_{n\to\infty} F_n(0) = \lim_{n\to\infty} n^2 \cdot 0 = 0.$$

Thus, $\{F_n\}$ converges pointwise to the zero function $F(x) \equiv 0$ on the interval $[0, 1]$. For each $n \ge 1$, we have

$$\begin{aligned} \int_0^1 F_n(x)\,dx &= \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (2n - n^2 x)\,dx + \int_{2/n}^1 0\,dx \\ &= \frac{n^2 x^2}{2}\bigg|_0^{\frac{1}{n}} + \left(2nx - \frac{n^2 x^2}{2}\right)\bigg|_{\frac{1}{n}}^{\frac{2}{n}} + 0 \\ &= \frac{1}{2} + (4 - 2) - (2 - \frac{1}{2}) + 0 = 1. \end{aligned}$$

Thus we have

$$\lim_{n\to\infty}\int_0^1 F_n(x)\,dx = \lim_{n\to\infty} 1 = 1 \neq 0 = \int_0^1 F(x)\,dx, \quad \text{i.e.}$$

$$\lim_{n\to\infty}\int_0^1 F_n(x)\,dx \neq \int_0^1 \Big(\lim_{n\to\infty} F_n(x)\Big)\,dx.$$

Reason: At different x, $F_n(x)$ converges to $F(x)$ at different pace (more specifically, in the definition of pointwise convergence, the choice of N depends on both ϵ and x).

2. Uniform Convergence

We define a slightly different concept of convergence.

Definition 2.1. $\{F_n\}$ is said to *converge uniformly* to a function F on an interval I if for every $\epsilon > 0$, there exists an N (which depends only on ϵ) such that

$$|F_n(x) - F(x)| < \epsilon$$

for ALL $x \in I$ whenever $n > N$.

Definition 2.2. $\{F_n\}$ is said to *converge uniformly on I* if $\{F_n\}$ converges uniformly to some F on I.

Theorem 2.3. *If $\{F_n\}$ converges uniformly to F on I, then $\{F_n\}$ converges pointwise to F on I.*

Remark. The limiting function F is unique.

Two Criteria for Uniform Convergence of $\{F_n\}$.

The following theorem is useful (computationally) in determining whether a sequence of functions converges uniformly or not.

Theorem 2.4. *Suppose $\{F_n\}$ is a sequence of functions converging pointwise to a function F on an interval I, and let*

$$T_n = \sup_{x\in I} |F_n(x) - F(x)|.$$

Then $\{F_n\}$ converges uniformly to F on I if and only if $\lim_{n\to 0} T_n = 0$.

Proof. First we prove the 'only if' part. Suppose that $\{F_n\}$ converges uniformly to F on I. Then for any given $\epsilon > 0$, there exists N such that

$$|F_n(x) - F(x)| < \frac{\epsilon}{2} \quad \text{for all } n > N \text{ and } x \in I$$

$$\Rightarrow\ T_n = \sup_{x\in I} |F_n(x) - F(x)| \leq \frac{\epsilon}{2} < \epsilon \quad \text{for all } n > N$$

$$\Rightarrow\ |T_n - 0| = T_n < \epsilon \quad \text{for all } n > N.$$

Hence we have $\lim_{n\to 0} T_n = 0$.

Next we prove the 'if' part. Suppose that $\lim_{n\to 0} T_n = 0$. Then for any given $\epsilon > 0$, there exists N such that

$$|T_n - 0| = T_n < \epsilon \quad \text{for all } n > N$$

$$\Rightarrow \sup_{x\in I} |F_n(x) - F(x)| < \epsilon \quad \text{for all } n > N$$

$$\Rightarrow |F_n(x) - F(x)| < \epsilon \quad \text{for all } n > N \text{ and } x \in I.$$

Hence $\{F_n\}$ converges uniformly to F on I. This finishes the proof of the theorem. ☐

THEOREM 2.5 (Cauchy's Criterion). *A sequence of functions $\{F_n\}$ converges uniformly on an interval I if and only if given any $\epsilon > 0$, there exists a natural number N such that*

$$|F_n(x) - F_m(x)| < \epsilon \quad \textit{for all } x \in I \textit{ and all } m, n > N. \tag{5}$$

Remark: Here N does not depend on x.

PROOF. First we prove the 'only if' part. Suppose that $\{F_n\}$ converges uniformly to the function F on I. Then given any $\epsilon > 0$, there exists N such that

$$|F_n(x) - F(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in I \text{ and all } n > N.$$

Then for all $x \in I$ and $m, n > N$,

$$\begin{aligned} |F_n(x) - F_m(x)| &= |(F_n(x) - F(x)) - (F_m(x) - F(x))| \\ &\le |F_n(x) - F(x)| + |F_m(x) - F(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This finishes the proof of the 'only if' part.

Next we prove the 'if' part. Suppose that equation 5 holds. Then for each fixed point $x \in I$, $\{F_n(x)\}$ is a Cauchy sequence of real numbers, and thus by Cauchy's criterion for sequences, the sequence of real numbers $\{F_n(x)\}$ converges. For each $x \in I$, we denote the limit by $F(x) = \lim_{n\to\infty} F_n(x)$. Then $\{F(x)\}_{x\in I}$ forms a function on I, which we denote by F. Given any $\epsilon > 0$, by equation 5, there exists N such that

$$|F_n(x) - F_m(x)| < \frac{\epsilon}{2} \quad \text{for all } x \in I \text{ and all } m, n > N.$$

Then for each fixed $x \in I$ and $n > N$, we have

$$\begin{aligned} |F_n(x) - F(x)| &= |F_n(x) - \lim_{m\to\infty} F_m(x)| \\ &= \lim_{m\to\infty} |F_n(x) - F_m(x)| \le \lim_{m\to\infty} \frac{\epsilon}{2} = \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Thus $\{F_n\}$ converges uniformly to F, and this finishes the proof of the 'if' part. ☐

EXAMPLE 2.6. *Show that $F_n(x) = \dfrac{\sin^2 x}{n}$, $\quad x \in (-\infty, +\infty)$, converges uniformly.*

PROOF. The limiting function $F(x)$ is

$$F(x) = \lim_{n\to\infty} \frac{\sin^2 x}{n} = 0$$

for all $x \in (-\infty, +\infty)$. Since

$$T_n = \sup_{x\in(-\infty,+\infty)} |F_n(x) - F(x)| = \sup_{x\in(-\infty,+\infty)} \left|\frac{\sin^2 x}{n}\right| \le \frac{1}{n} \to 0$$

as $n \to \infty$, thus the sequence of functions $\{F_n(x\}$ converges uniformly on $(-\infty, +\infty)$. ☐

EXAMPLE 2.7. *Determine whether the following sequences of functions converge uniformly on the indicated interval.*

(a) $f_n(x) = \dfrac{n^2 \ln x}{x^n}$, $x \in [1, +\infty)$;

(b) $f_n(x) = \dfrac{n^2 \ln x}{x^n}$, $x \in [2, +\infty)$.

SOLUTION. Let $f(x) = \lim\limits_{n\to\infty} f_n(x) = 0$ for $x \geq 1$.

(a). $T_n = \sup\limits_{x\geq 1} |f_n(x) - 0| = \sup\limits_{x\geq 1} \dfrac{n^2 \ln x}{x^n} = \sup\limits_{x\geq 1} f_n(x)$. From

$$f_n'(x) = n^2 \frac{1}{x} \cdot x^{-n} - n^3 \ln x \cdot x^{-n-1} = \frac{n^2 - n^3 \ln x}{x^{n+1}} = 0,$$

we have $n^2 - n^3 \ln x = 0$ or $x = e^{\frac{1}{n}}$. Observe that $f_n(x)$ is monotone increasing for $1 \leq x \leq e^{\frac{1}{n}}$ and monotone decreasing for $x \geq e^{\frac{1}{n}}$. Thus

$$T_n = \max_{x\geq 1} f_n(x) = f_n(e^{\frac{1}{n}}) = \frac{n^2 \cdot \frac{1}{n}}{\left(e^{\frac{1}{n}}\right)^n} = \frac{n}{e} \not\to 0$$

as $n \to \infty$ and so $\{f_n(x)\}$ does NOT converge uniformly.

(b). Since $e^{\frac{1}{n}} \leq 2$ for $n \geq 2$, the function $f_n(x)$ is monotone decreasing on $[2, +\infty)$ for $n \geq 2$ and so $T_n = \sup\limits_{x\geq 2} |f_n(x) - f(x)| = f_n(2) = \dfrac{n^2 \ln 2}{2^n}$ for $n \geq 2$. Since $\lim\limits_{n\to\infty} T_n = 0$, $\{f_n\}$ converges uniformly on $[2, +\infty)$. □

EXAMPLE 2.8. *Show that* $F_n(x) = \dfrac{n^2 \ln x \sin nx}{x^n}$ *converges uniformly on* $[2, +\infty)$.

SOLUTION. $F(x) = \lim\limits_{n\to\infty} F_n(x) = 0$ for $x \geq 2$. Observe

$$T_n = \sup_{x\geq 2} |F_n(x) - F(x)| = \sup_{x\geq 2} \frac{n^2 \ln x |\sin nx|}{x^n} \leq \frac{n^2 \ln 2}{2^n}$$

for $n \geq 2$. Since $\lim\limits_{n\to\infty} T_n = 0$, $\{F_n\}$ converges uniformly. □

Remark. Let $\{F_n\}$ be a sequence of functions on an interval I. To see whether $\{F_n\}$ is uniformly convergent, we may try to do by the following steps.

(1). Determine the limiting function $F(x) = \lim\limits_{n\to\infty} F_n(x)$.

(2). Determine $T_n = \sup\limits_{x\in I} |F_n(x) - F(x)|$.

(3). Check whether $\lim\limits_{n\to\infty} T_n = 0$.

If T_n is difficult to be determined, then we may try to estimate an upper bound of T_n (a lower bound of T_n if we guess that the sequence of functions might not be uniformly convergent).

3. Importance of Uniform Convergence of $\{F_n\}$

We address the 2 questions raised in section 1 of this chapter in the context of uniform convergence.

Suppose a sequence of continuous functions $\{F_n\}$ converges uniformly to a function F.

1. Will F be continuous?
2. Is $\lim\limits_{n\to\infty} \int_a^b F_n(x)\,dx = \int_a^b F(x)\,dx$?

The answer to both questions is YES.

Question 1 will be answered by the following theorem, while Question 2 will be answered later (by Theorem 4.2).

THEOREM 3.1. *Let $\{F_n\}$ be a sequence of continuous functions on an interval I. Suppose that $\{F_n\}$ converges uniformly to a function F on I. Then F is continuous on I.*

PROOF. Fix any point $x_0 \in I$. We are going to show that F is continuous at the point x_0. Given any $\epsilon > 0$, since $\{F_n\}$ converges uniformly to F on I, there exists N such that

$$|F_n(x) - F(x)| < \frac{\epsilon}{3} \quad \text{for all } x \in I \text{ and all } n > N.$$

Next we fix an $n > N$ (say, $n = [N] + 1$). Since F_n is continuous at x_0, there exists $\delta > 0$ (here δ depends on x_0 and ϵ) such that for all x satisfying $|x - x_0| < \delta$, we have

$$|F_n(x) - F_n(x_0)| < \frac{\epsilon}{3}.$$

Then for all x satisfying $|x - x_0| < \delta$, we have

$$\begin{aligned} |F(x) - F(x_0)| &= |F(x) - F_n(x) + F_n(x) - F_n(x_0) + F_n(x_0) - F(x_0)| \\ &\le |F(x) - F_n(x)| + |F_n(x) - F_n(x_0)| + |F_n(x_0) - F(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus F is continuous at x_0. Since x_0 is arbitrary, it follows that F is continuous on I. This finishes the proof of the theorem. □

EXAMPLE 3.2. *Find the pointwise limit F of the sequence*

$$F_n(x) = \frac{x^{2n}}{1 + x^{2n}}, \qquad x \in [0, 1].$$

Show using Theorem 3.1 that the convergence is not uniform.

SOLUTION. If $0 \le x < 1$, we have

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{x^{2n}}{1 + x^{2n}} = \frac{0}{1+0} = 0.$$

If $x = 1$, then $F_n(1) = \frac{1}{2}$. Thus the limiting function $F(x)$ is

$$F(x) = \begin{cases} 0 & 0 \le x < 1 \\ \frac{1}{2} & x = 1 \end{cases}$$

Because each $F_n(x)$ is continuous but $F(x)$ is not, the sequence of functions $\{F_n(x)\}$ does not converge uniformly on $[0,1]$ by Theorem 3.1. $\square$

4. Uniform Convergence and Riemann Integration

Before we go on, we first recall some facts about Riemann integrals (Reference: Chapter 6, p.150-160 of [1]).

Review of Riemann Integration

Let f be a bounded function on a finite interval $[a,b]$. A *partition* P of $[a,b]$ is a set of points $\{x_0, x_1, \cdots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For such a partition P and $i = 1, 2, \cdots n$, we denote

$$M_i(f) = \sup_{x\in[x_{i-1},x_i]} f(x)$$

$$m_i(f) = \inf_{x\in[x_{i-1},x_i]} f(x).$$

The *upper (Riemann) sum* of f with respect to the partition P is defined to be

$$U(P,f) := \sum_{i=1}^{n} M_i(f)\Delta x_i.$$

Here $\Delta x_i = x_i - x_{i-1}$. Similarly the *lower (Riemann) sum* of f with respect to P is defined to be

$$L(P,f) := \sum_{i=1}^{n} m_i(f)\Delta x_i.$$

Then f is said to be *Riemann integrable* on $[a,b]$ if and only if

$$\inf_P U(P,f) = \sup_P L(P,f),$$

where the infimum and supremum are taken over all partitions P of $[a,b]$. When f is Riemann integrable, the common value of the above equality is called the *Riemann integral* of f over $[a,b]$, and it is denoted by $\displaystyle\int_a^b f(x)\,dx$.

THEOREM 4.1. *A function f is Riemann integrable on a finite interval $[a,b]$ if and only if given any $\epsilon > 0$, there exists a partition P of $[a,b]$ such that*

$$U(P,f) - L(P,f) < \epsilon.$$

Remark. Any continuous function on a finite interval $[a,b]$ is Riemann integrable.

– End of Review –

THEOREM 4.2. *Let $\{F_n\}$ be a sequence of Riemann integrable (bounded) functions on a finite interval $[a,b]$. Suppose that $\{F_n\}$ converges uniformly to a function F on $[a,b]$. Then F is Riemann integrable on $[a,b]$, and*

$$\lim_{n\to\infty}\int_a^b F_n(x)\,dx = \int_a^b F(x)\,dx \quad i.e. \quad \lim_{n\to\infty}\int_a^b F_n(x)\,dx = \int_a^b \Big(\lim_{n\to\infty} F_n(x)\Big)\,dx.$$

Proof. First we show that $F(x)$ is a bounded function. Since $\{F_n\}$ converges uniformly to F on $[a,b]$, for $\epsilon = 1$, there exists N such that

$$|F_n(x) - F(x)| < 1$$

for all $n > N$ and all $x \in [a,b]$. In particular,

$$|F_{N+1}(x) - F(x)| < 1 \quad \text{or} \quad F_{N+1}(x) - 1 < F(x) < F_{N+1}(x) + 1$$

for all $x \in [a,b]$. By the assumption, each F_n is bounded function and so there exists numbers m and M such that

$$m \leq F_{N+1}(x) \leq M$$

for all $x \in [a,b]$. Now, for all $x \in [a,b]$, we have

$$m - 1 \leq F_{N+1}(x) - 1 < F(x) < F_{N+1}(x) + 1 \leq M + 1$$

and so $F(x)$ is bounded.

Next we show that F is Riemann integrable on $[a,b]$ using the criterion in Theorem 4.1. Given any $\epsilon > 0$, since $\{F_n\}$ converges uniformly to F on $[a,b]$, there exists N such that

$$|F_n(x) - F(x)| < \frac{\epsilon}{3(b-a)} \quad \text{for all } x \in [a,b] \text{ and all } n > N. \tag{6}$$

Fix an $n > N$ (say, $n = [N] + 1$). Since F_n is Riemann integrable on $[a,b]$, it follows from Theorem 4.1 that there exists a partition P of $[a,b]$ such that

$$U(P, F_n) - L(P, F_n) < \frac{\epsilon}{3}.$$

Fix such a partition P, and write

$$M_i(F) = \sup_{x \in [x_{i-1}, x_i]} F(x), \qquad M_i(F_n) = \sup_{x \in [x_{i-1}, x_i]} F_n(x),$$

$$m_i(F) = \inf_{x \in [x_{i-1}, x_i]} F(x), \qquad m_i(F_n) = \inf_{x \in [x_{i-1}, x_i]} F_n(x).$$

From Equation 6, we have, for $i = 1, 2, \cdots, n$,

$$|M_i(F) - M_i(F_n)| \leq \frac{\epsilon}{3(b-a)},$$

$$|m_i(F) - m_i(F_n)| \leq \frac{\epsilon}{3(b-a)}.$$

Then

$$U(P,F) - L(P,F) = \sum_{i=1}^n M_i(F)\Delta x_i - \sum_{i=1}^n m_i(F)\Delta x_i = \sum_{i=1}^n \big(M_i(F) - m_i(F)\big)\Delta x_i$$

$$= \sum_{i=1}^n \big[M_i(F) - M_i(F_n) + M_i(F_n) - m_i(F_n) + m_i(F_n) - m_i(F)\big]\Delta x_i$$

$$\leq \sum_{i=1}^n \big[|M_i(F) - M_i(F_n)| + M_i(F_n) - m_i(F_n) + |m_i(F_n) - m_i(F)|\big]\Delta x_i$$

$$= \sum_{i=1}^n |M_i(F) - M_i(F_n)|\Delta x_i + \sum_{i=1}^n M_i(F_n)\Delta x_i - \sum_{i=1}^n m_i(F_n)\Delta x_i + \sum_{i=1}^n |m_i(F_n) - m_i(F)|\Delta x_i$$

$$
\begin{aligned}
&= U(P, F_n) - L(P, F_n) + \sum_{i=1}^{n} |M_i(F) - M_i(F_n)|\Delta x_i + \sum_{i=1}^{n} |m_i(F_n) - m_i(F)|\Delta x_i \\
&< \frac{\epsilon}{3} + \sum_{i=1}^{n} \frac{\epsilon}{3(b-a)}\Delta x_i + \sum_{i=1}^{n} \frac{\epsilon}{3(b-a)}\Delta x_i \\
&= \frac{\epsilon}{3} + \frac{2\epsilon}{3(b-a)} \sum_{i=1}^{n} \Delta x_i = \frac{\epsilon}{3} + \frac{2\epsilon}{3(b-a)} \cdot (b-a) = \epsilon.
\end{aligned}
$$

Using the criterion in Theorem 4.1, it follows that F is Riemann integrable on $[a, b]$.

Now we are going to prove the equation in the theorem. For any given $\epsilon > 0$, since $\{F_n\}$ converges uniformly to F on $[a, b]$, there exists N such that

$$
|F_n(x) - F(x)| < \frac{\epsilon}{2(b-a)} \quad \text{for all } x \in [a, b] \text{ and all } n > N.
$$

Then

$$
\left|\int_a^b F_n(x)\,dx - \int_a^b F(x)\,dx\right| = \left|\int_a^b \big(F_n(x) - F(x)\big)\,dx\right|
$$

$$
\le \int_a^b |F_n(x) - F(x)|\,dx \le \int_a^b \frac{\epsilon}{2(b-a)}\,dx \le \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2} < \epsilon.
$$

Thus we have $\lim_{n\to\infty} \int_a^b F_n(x)\,dx = \int_a^b F(x)\,dx$, and this finishes the proof of the theorem. ☐

Example 4.3. *Compute, justifying your answer,*

$$
\lim_{n\to\infty} \int_0^1 \frac{\sin nx}{n + x^2}\,dx.
$$

Solution. Let $F_n(x) = \dfrac{\sin nx}{n + x^2}$. Then the limiting function

$$
F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{\sin nx}{n + x^2} = 0
$$

for any given $0 \le x \le 1$, by the Squeeze theorem, because

$$
-\frac{1}{n} \le \frac{\sin nx}{n + x^2} \le \frac{1}{n}
$$

and $\lim_{n\to\infty} \dfrac{1}{n} = -\lim_{n\to\infty} \dfrac{1}{n} = 0$. Since

$$
T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} \left|\frac{\sin nx}{n + x^2}\right| \le \frac{1}{n} \to 0
$$

as $n \to \infty$, the sequence of functions $\{F_n\}$ converges uniformly to $F(x)$ and so, by Theorem 4.2,

$$
\lim_{n\to\infty} \int_0^1 \frac{\sin nx}{n + x^2}\,dx = \int_0^1 \lim_{n\to\infty} \frac{\sin nx}{n + x^2}\,dx = \int_0^1 0\,dx = 0.
$$

☐

5. Series of Functions

A *series of functions* on an interval I is of the form

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \cdots ,$$

where each f_n is a function on I.

Example 5.1. Below are some examples.

1. $\displaystyle\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \cdots$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k+x} = \frac{\sin x}{1+x} + \frac{\sin 2x}{2+x} + \frac{\sin 3x}{3+x} + \cdots , \qquad 0 \le x \le 1.$

As in chapter 2, we may form the partial sums

$$S_n(x) = \sum_{k=1}^{n} f_k(x) = f_1(x) + f_2(x) + \cdots + f_n(x).$$

Then $\{S_n\}$ forms a sequence of functions on I.

Definition 5.2. The series $\displaystyle\sum_{n=1}^{\infty} f_n$ is said to *converge pointwise* (to a function S) on I if $\{S_n\}$ converges pointwise (to S) on I, (i.e. $\displaystyle\lim_{n\to\infty} S_n(x) = S(x)$ for each $x \in I$.)

Example 5.3. *What is the pointwise limit of* $\displaystyle\sum_{n=1}^{\infty} x^{n-1}$*, where* $x \in (-1, 1)$*?*

Solution. Consider the partial sum

$$S_n(x) = \sum_{i=1}^{n} x^{i-1} = 1 + x + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$$

for $-1 < x < 1$. Thus

$$\sum_{n=1}^{\infty} x^{n-1} = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}$$

for $x \in (-1, 1)$. □

Definition 5.4. $\displaystyle\sum_{n=1}^{\infty} f_n$ is said to *converge uniformly (to S)* on I if $\{S_n\}$ converges uniformly (to S) on I.

Example 5.5. *Does* $\displaystyle\sum_{n=1}^{\infty} x^{n-1}$ *converge uniformly on* $[0, \frac{1}{2}]$*?*

SOLUTION. Let $S(x) = \sum_{n=1}^{\infty} x^{n-1}$. Then $S(x) = \frac{1}{1-x}$ for $x \in (-1, 1)$ by the above example. Now

$$T_n = \sup_{0 \le x \le \frac{1}{2}} |S_n(x) - S(x)| = \sup_{0 \le x \le \frac{1}{2}} \left| \frac{1-x^n}{1-x} - \frac{1}{1-x} \right| = \sup_{0 \le x \le \frac{1}{2}} \left| \frac{-x^n}{1-x} \right|$$

$$= \sup_{0 \le x \le \frac{1}{2}} \frac{x^n}{1-x} \le \frac{\left(\frac{1}{2}\right)^n}{\frac{1}{2}} = \left(\frac{1}{2}\right)^{n-1}$$

because $x^n \le \left(\frac{1}{2}\right)^n$ and $1 - x \ge \frac{1}{2}$ for $0 \le x \le \frac{1}{2}$. Since $\lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so the sequence of functions $\{S_n\}$ converges uniformly to $S(x)$ on $[0, \frac{1}{2}]$. Thus the series of functions $\sum_{n=1}^{\infty} x^{n-1}$ converges uniformly on $[0, \frac{1}{2}]$.

$\square$

The following test is very useful in verifying that certain series of functions converge uniformly to some functions on an interval.

THEOREM 5.6 (Weierstrass M-test). *Consider a series of functions* $\sum_{k=1}^{\infty} f_k$ *on an interval* I. *Suppose that*

(i) $|f_k(x)| \le M_k$ *for all* $x \in I$, $k = 1, 2, \cdots$, *and*

(ii) $\sum_{k=1}^{\infty} M_k$ *converges.*

Then $\sum_{k=1}^{\infty} f_k$ *converges uniformly (to some function) on* I.

PROOF. For $n = 1, 2, \cdots$, let $S_n(x) = \sum_{k=1}^{n} f_k(x)$, $x \in I$. Since $\sum_{k=1}^{\infty} M_k$ converges (by (ii)), the sequence of partial sums $\{s_n = \sum_{k=1}^{n} M_k\}$ converges, and is thus a Cauchy sequence of real numbers by the Cauchy Criterion. Thus, given any $\epsilon > 0$, there exists N such that

$$|s_n - s_m| < \epsilon \quad \text{for all } n > m > N$$

$$\Rightarrow |\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k| < \epsilon \quad \text{for all } n > m > N$$

$$\Rightarrow \sum_{k=m+1}^{n} M_k < \epsilon \quad \text{for all } n > m > N.$$

Then for all $x \in I$, we have

$$\begin{aligned}|S_n(x) - S_m(x)| &= \Big|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\Big| = \Big|\sum_{k=m+1}^{n} f_k(x)\Big| \\ &\le \sum_{k=m+1}^{n} |f_k(x)| \le \sum_{k=m+1}^{n} M_k < \epsilon.\end{aligned}$$

Thus by the Cauchy criterion for functions, $\{S_n\}$ converges uniformly (to some function) on I, i.e. $\sum\limits_{k=1}^{\infty} f_k$ converges uniformly (to that function) on I. ☐

Example 5.7. *Show that* $\sum\limits_{n=1}^{\infty} \frac{\cos^n x}{n^2 + x}$ *converges uniformly on* $(0, \infty)$.

Proof. Since

$$\left|\frac{\cos^n x}{n^2 + x}\right| \le \frac{1}{n^2}$$

for all $x \in (0, \infty)$ and the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series, the series of functions $\sum\limits_{n=1}^{\infty} \frac{\cos^n x}{n^2 + x}$ converges uniformly by the Weierstrass M-test and hence the result. ☐

Example 5.8. *Show that the series* $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ *does not converge uniformly on* $[0, 1)$.

Proof. Observe that

$$T_n = \sup_{0 \le x < 1} |S_n(x) - S(x)| = \sup_{0 \le x < 1} \left|\sum_{k=n+1}^{\infty} \frac{x^k}{k}\right| = \sup_{0 < x < 1} \sum_{k=n+1}^{\infty} \frac{x^k}{k}.$$

We have

$$\begin{aligned}T_{2^s} &= \sup_{0<x<1} \sum_{k=2^s+1}^{\infty} \frac{x^k}{k} \ge \sup_{0<x<1} \sum_{k=2^s+1}^{2^{s+1}} \frac{x^k}{k} \\ &= \sup_{0<x<1} \left(\frac{x^{2^s+1}}{2^s+1} + \frac{x^{2^s+2}}{2^s+2} + \cdots + \frac{x^{2^{s+1}}}{2^{s+1}}\right) \\ &\ge \sup_{0<x<1} \left(\frac{x^{2^s+1}}{2^{s+1}} + \frac{x^{2^s+2}}{2^{s+1}} + \cdots + \frac{x^{2^{s+1}}}{2^{s+1}}\right) \\ &= \sup_{0<x<1} \frac{x^{2^s+1}}{2^{s+1}} \left(1 + x + x^2 + \cdots + x^{2^s-1}\right) \ge \frac{1}{2}\end{aligned}$$

because

$$\lim_{x \to 1} \frac{x^{2^s+1}}{2^{s+1}} \left(1 + x + x^2 + \cdots + x^{2^s-1}\right) = \frac{1}{2^{s+1}}(1 + 1 + \cdots + 1) = \frac{1}{2^{s+1}} \cdot 2^s = \frac{1}{2}.$$

Thus the subsequence $\{T_{2^s}\}$ does not tend to 0 and so the sequence $\{T_n\}$ does not tend to 0. It follows that the series of functions $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$ does NOT converge uniformly on $[0, 1)$.

□

6. Importance of Uniform Convergence for Series of Functions

Corollary 6.1. *Suppose that* $\displaystyle\sum_{k=1}^{\infty} f_k$ *converges uniformly to a function* S *on an interval* I*. Suppose that each* f_k *is continuous on* I*. Then* S *is also continuous on* I*.*

Proof. Consider the sequence of partial sums $\{S_n\}$ on I, where we have $S_n = \displaystyle\sum_{k=1}^{n} f_k$. Then $\{S_n\}$ converges uniformly to S on I. If each f_k is continuous on I, then each S_n is also continuous on I. Then by Theorem 3.1, S is also continuous on I. □

Example 6.2. *Is* $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n^2 e^{nx}}, \quad x \in (0, \infty)$, *a continuous function?*

Solution. Let $f_n(x) = \dfrac{x}{n^2 e^{nx}}$. We find an upper bound of $f_n(x)$. Observe that

$$f_n'(x) = \left(\frac{x}{n^2 e^{nx}}\right)' = \left(\frac{x e^{-nx}}{n^2}\right)' = \frac{e^{-nx} - nxe^{-nx}}{n^2} = \frac{e^{-nx}\left(\dfrac{1}{n} - x\right)}{n}.$$

We obtain that $f_n'(x) > 0$ for $0 < x < \frac{1}{n}$ and $f_n'(x) < 0$ for $x > \frac{1}{n}$. It follows that $f_n(x)$ is monotone increasing on $(0, \frac{1}{n}]$ and monotone decreasing on $[\frac{1}{n}, +\infty)$. Thus

$$\sup_{0<x<+\infty} f_n(x) = \max_{0<x<+\infty} f_n(x) = f_n\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{n^2 e^{n \cdot \frac{1}{n}}} = \frac{1}{en^3}.$$

Let $M_n = \dfrac{1}{en^3}$. Then $|f_n(x)| \le M_n$ for $x \in (0, \infty)$. Since $\displaystyle\sum_{n=1}^{\infty} M_n$ converges by the p-series, the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n^2 e^{nx}}$ converges uniformly by Weierstrass M-test on $(0, \infty)$. According to Corollary 6.1, the function $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n^2 e^{nx}}$ is continuous on $(0, \infty)$.

□

Corollary 6.3. *Suppose that* $\displaystyle\sum_{k=1}^{\infty} f_k$ *converges uniformly to a function* S *on an interval* $[a, b]$*. Suppose that each* f_k *is a Riemann integrable (bounded) function on* $[a, b]$*. Then* S *is also Riemann integrable on* $[a, b]$*, and*

$$\int_a^b S(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx, \quad \textit{i.e.} \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx.$$

Proof. Consider the sequence of partial sums $\{S_n\}$ on $[a,b]$, where $S_n = \sum_{k=1}^{n} f_k$. Then $\{S_n\}$ converges uniformly to S on $[a,b]$. If each f_k is Riemann integrable on $[a,b]$, then each S_n is also Riemann integrable on $[a,b]$. Then by Theorem 4.2, S is also Riemann integrable on $[a,b]$, and

$$\begin{aligned}\int_a^b S(x)\,dx &= \lim_{n\to\infty}\int_a^b S_n(x)\,dx = \lim_{n\to\infty}\int_a^b \sum_{k=1}^{n} f_k(x)\,dx\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_a^b f_k(x)\,dx = \sum_{k=1}^{\infty}\int_a^b f_k(x)\,dx.\end{aligned}$$

□

By using this theorem, we have the following amazing formula.

Example 6.4. *Show that*

$$\ln 2 = \sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n} = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \cdots$$

Proof. According to Example 5.5, the series $\sum_{n=1}^{\infty} x^{n-1}$ converges uniformly to $\frac{1}{1-x}$ on $[0,\frac{1}{2}]$. Thus we have

$$\begin{aligned}\int_0^{\frac{1}{2}}\frac{1}{1-x} &= \int_0^{\frac{1}{2}}\sum_{n=1}^{\infty} x^{n-1}\,dx\\ &= \sum_{n=1}^{\infty}\int_0^{\frac{1}{2}} x^{n-1}\,dx = \sum_{n=1}^{\infty}\frac{x^n}{n}\bigg|_0^{1/2} = \sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}.\end{aligned}$$

Since

$$\int_0^{\frac{1}{2}}\frac{1}{1-x} = -\ln(1-x)\Big|_0^{\frac{1}{2}} = -\ln\left(1-\frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right) = \ln 2,$$

we obtain the formula

$$\ln 2 = \sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n} = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \cdots.$$

□

Remark. Example 6.4 gives a way to estimate the number ln 2 because the remainder

$$\begin{aligned}R_n = \sum_{k=n+1}^{\infty}\frac{1}{k\cdot 2^k} &= \frac{1}{(n+1)2^{n+1}} + \frac{1}{(n+2)2^{n+2}} + \cdots\\ &< \frac{1}{(n+1)2^{n+1}} + \frac{1}{(n+1)2^{n+2}} + \frac{1}{(n+1)2^{n+3}} + \cdots\\ &= \frac{1}{(n+1)2^{n+1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) = \frac{1}{(n+1)2^{n+1}}\cdot\frac{1}{1-\frac{1}{2}} = \frac{1}{(n+1)2^n}.\end{aligned}$$

For instance,

$$\ln 2 \approx \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots + \frac{1}{10 \cdot 2^{10}}$$

with error less than $\dfrac{1}{11 \cdot 2^{10}} = \dfrac{1}{11264}$.

THEOREM 6.5. *Let $\{F_n\}$ be a sequence of functions on $[a,b]$ such that*

(i) *each F_n' exists and is continuous on $[a,b]$,*
(ii) *$\{F_n\}$ converges pointwise to a function F on $[a,b]$, and*
(iii) *$\{F_n'\}$ converges uniformly on $[a,b]$.*

Then F is differentiable on $[a,b]$, and for all $x \in [a,b]$,

$$F'(x) = \lim_{n\to\infty} F_n'(x), \ \ i.e. \frac{d}{dx}\big(\lim_{n\to\infty} F_n(x)\big) = \lim_{n\to\infty}\big(\frac{d}{dx}F_n(x)\big).$$

Remark. Here the differentiability and continuity at the endpoints a and b refer to the one sided derivatives and limits respectively.

PROOF. By (iii), there exists a function g such that $\{F_n'\}$ converges uniformly to g on $[a,b]$. In particular, $\lim\limits_{n\to\infty} F_n'(x) = g(x)$ for all $x \in [a,b]$. By (i), since each F_n' is continuous on $[a,b]$, F_n' is also Riemann integrable on $[a,b]$, and by the fundamental theorem of calculus,

$$\int_a^x F_n'(t)\,dt = F_n(x) - F_n(a) \quad \text{for all } x \in [a,b].$$

Letting $n \to \infty$, we have, for all $x \in [a,b]$,

$$\lim_{n\to\infty}\int_a^x F_n'(t)\,dt = \lim_{n\to\infty}\big(F_n(x) - F_n(a)\big) = F(x) - F(a). \tag{7}$$

On the other hand, since $\{F_n'\}$ converges uniformly to g on $[a,b]$, it follows from Theorem 4.2 that

$$\lim_{n\to\infty}\int_a^x F_n'(t)\,dt = \int_a^x \big(\lim_{n\to\infty} F_n'(t)\big)dt = \int_a^x g(t)\,dt.$$

Together with equation 7, it follows that

$$F(x) - F(a) = \int_a^x g(t)\,dt \quad \text{for all } x \in [a,b].$$

By (i) and Theorem 3.1, g is continuous on $[a,b]$. Then by the fundamental theorem of calculus, we have

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x).$$

Together with equation 7, it follows that F is also differentiable on $[a,b]$, and for all $x \in [a,b]$,

$$\frac{d}{dx}\big(F(x) - F(a)\big) = \frac{d}{dx}\int_a^x g(t)\,dt = g(x), \quad \text{i.e.} F'(x) = g(x), \quad \text{i.e.}$$

$$\frac{d}{dx}\big(\lim_{n\to\infty} F_n(x)\big) = \lim_{n\to\infty}\big(\frac{d}{dx}F_n(x)\big).$$

The proof is finished. □

REMARK 6.6. *By inspecting the proof, Theorem 6.5 still holds when the closed interval* $[a,b]$ *is replaced by* (a,b), $(a,b]$ *or* $[a,b)$.

COROLLARY 6.7. *Let* $\sum_{k=1}^{\infty} f_k$ *be a series of functions on* $[a,b]$ *such that*

(i) *each* f_k' *exists and is continuous on* $[a,b]$,

(ii) $\sum_{k=1}^{\infty} f_k$ *converges pointwise to a function* S *on* $[a,b]$, *and*

(iii) $\sum_{k=1}^{\infty} f_k'$ *converges uniformly on* $[a,b]$.

Then S *is differentiable on* $[a,b]$, *and for all* $x \in [a,b]$,

$$S'(x) = \sum_{k=1}^{\infty} f_k'(x), \quad \textit{i.e.} \ \frac{d}{dx}\Big(\sum_{k=1}^{\infty} f_k(x)\Big) = \sum_{k=1}^{\infty} \frac{d}{dx} f_k(x).$$

PROOF. Consider the sequence of partial sums (of functions) $\{S_n\}$ on $[a,b]$, where $S_n = \sum_{k=1}^{n} f_k$ for each n. Then by (ii),

$$\lim_{n\to\infty} S_n(x) = S(x) \quad \text{for all } x \in [a,b].$$

The conditions (i), (ii), (iii) of the corollary imply that the sequence of functions $\{S_n\}$ satisfies the corresponding conditions (i), (ii), (iii) of Theorem 6.5. Thus by Theorem 6.5, the function S is differentiable on $[a,b]$, and for all $x \in [a,b]$,

$$S'(x) = \lim_{n\to\infty} S_n'(x), \quad \text{i.e.} \frac{d}{dx}\Big(\sum_{k=1}^{\infty} f_k(x)\Big)$$

$$= \lim_{n\to\infty} \Big(\sum_{k=1}^{n} f_k(x)\Big)' = \lim_{n\to\infty} \sum_{k=1}^{n} f_k'(x) = \sum_{k=1}^{\infty} \frac{d}{dx} f_k(x).$$

□

REMARK 6.8. *According to Remark 6.6, Corollary 6.7 still holds when the closed interval* $[a,b]$ *is replaced by* (a,b), $(a,b]$ *or* $[a,b)$.

7. Power Series

DEFINITION 7.1. A *power series* in x is of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

EXAMPLE 7.2. Below are some examples

1. $\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$
2. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

DEFINITION 7.3. A *power series* in $x - x_0$ is of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

EXAMPLE 7.4. Here are some examples.

1. $\displaystyle\sum_{n=0}^{\infty}(x-1)^n = 1 + (x-1) + (x-1)^2 + \cdots$.
2. $\displaystyle\sum_{n=1}^{\infty} n^2(x+2)^n = (x+2) + 2^2(x+2)^2 + 3^2(x+2)^3 + \cdots$.

Warning. Don't expand out the terms $a_n(x-x_0)^n$ in the power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$ because, when you rearrange terms in an (infinite) series, you may get different values. (For partial sums, you can expand out, if it is necessary, because there are only finitely many terms.)

Question: Given a power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-x_0)^n$, when does it converge and when does it diverge?

THEOREM 7.5. *Given any power series* $\displaystyle\sum_{k=0}^{\infty} a_k(x-x_0)^k$, *there is an associated number* R, $0 \le R \le \infty$, *called the* radius of convergence, *with the following properties:*

(i) *The series of real numbers* $\displaystyle\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *converges absolutely at each point* x *satisfying* $|x-x_0| < R$.

(ii) $\displaystyle\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *diverges at each* x *satisfying* $|x-x_0| > R$.

(iii) *The series of functions* $\displaystyle\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *converges uniformly on the interval* $|x-x_0| \le \rho$ *for any* ρ *satisfying* $0 < \rho < R$.

Moreover, R *is given by*

$$R = \frac{1}{\limsup |a_k|^{\frac{1}{k}}}. \tag{8}$$

($R = 0$ *if* $\limsup |a_k|^{\frac{1}{k}} = \infty$*; and* $R = \infty$ *if* $\limsup |a_k|^{\frac{1}{k}} = 0$*.) In addition, if* $\displaystyle\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ *exists, then* R *is also given by*

$$R = \frac{1}{\displaystyle\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}}. \tag{9}$$

PROOF. Let $R = \dfrac{1}{\limsup |a_k|^{\frac{1}{k}}}$ be as given in Equation 8. First we are going to prove (i). For each point x satisfying $|x - x_0| < R$, we consider the series of (non-negative) real numbers $\sum_{k=0}^{\infty} |a_k(x - x_0)^k|$. We have

$$\limsup \left(\left|a_k(x-x_0)^k\right|\right)^{\frac{1}{k}} = \limsup \left(|a_k|^{\frac{1}{k}}|x-x_0|\right) = |x-x_0|\cdot\limsup |a_k|^{\frac{1}{k}} < R\cdot\frac{1}{R} = 1.$$

Thus by the root test, $\sum_{k=0}^{\infty} |a_k(x - x_0)^k|$ converges, i.e. $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ converges absolutely.

Next we are going to prove (ii) by contradiction. Suppose that $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ converges at a point x satisfying $|x - x_0| > R$. Then by Theorem 1.7, we have

$$\lim_{k\to\infty} a_k(x - x_0)^k = 0.$$

Letting $\epsilon = 1$. Then there exists N such that

$$|a_k(x - x_0)^k - 0| < 1 \quad \text{for all } k > N \Rightarrow \quad |a_k(x - x_0)^k|^{\frac{1}{k}} < 1 \quad \text{for all } k > N$$

$$\Rightarrow \quad |a_k|^{\frac{1}{k}} < \frac{1}{|x - x_0|} \quad \text{for all } k > N \Rightarrow \quad \sup_{n\geq k} |a_k|^{\frac{1}{k}} \leq \frac{1}{|x - x_0|} \quad \text{for all } n > N$$

$$\Rightarrow \quad \limsup |a_k|^{\frac{1}{k}} \leq \frac{1}{|x - x_0|} \Rightarrow \quad \frac{1}{R} \leq \frac{1}{|x - x_0|} < \frac{1}{R},$$

which is a contradiction. Hence we must have $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ diverges at each x satisfying $|x - x_0| > R$.

Now we are going to prove (iii). Let ρ be a number such that $0 < \rho < R$. First, we have, for each k and all x satisfying $|x - x_0| \leq \rho$,

$$|a_k(x - x_0)^k| \leq |a_k| \cdot \rho^k. \tag{10}$$

Now we apply the root test to the series $\sum_{k=0}^{\infty} |a_k| \cdot \rho^k$. We have

$$\limsup \left(|a_k| \cdot \rho^k\right)^{\frac{1}{k}} = \limsup \left[\left(|a_k|\right)^{\frac{1}{k}} \cdot \rho\right] = \rho \cdot \limsup |a_k|^{\frac{1}{k}} < R \cdot \frac{1}{R} = 1.$$

Thus by the root test, $\sum_{k=0}^{\infty} |a_k| \cdot \rho^k$ converges. Together with equation 10, it follows from the Weierstrass M-test that the series of functions $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ converges uniformly on $[x_0 - \rho, x_0 + \rho]$.

Finally the proof of the Theorem under the condition (9) in place of (8) is similar (with the root test replaced by the ratio test in various places), and it will be left to the student as an exercise. □

EXAMPLE 7.6. *What is the radius of convergence for the series*

$$1+\frac{x}{3}+\frac{x^2}{4^2}+\frac{x^3}{3^3}+\frac{x^4}{4^4}+\frac{x^5}{3^5}+\frac{x^6}{4^6}+\cdots$$

SOLUTION. Since

$$a_n=\begin{cases}\dfrac{1}{4^{2k}} & n=2k\\[2ex] \dfrac{1}{3^{2k-1}} & n=2k-1,\end{cases}$$

we have

$$\sqrt[n]{|a_n|}=\begin{cases}\dfrac{1}{4} & n=2k\\[2ex] \dfrac{1}{3} & n=2k-1.\end{cases}$$

Thus $\overline{\lim\limits_{n\to\infty}}\ \sqrt[n]{|a_n|}=\dfrac{1}{3}$ and so the radius of convergence

$$R=\frac{1}{\overline{\lim\limits_{n\to\infty}}\ \sqrt[n]{|a_n|}}=\frac{1}{\frac{1}{3}}=3.$$

□

COROLLARY 7.7. *Given any* $\sum\limits_{k=0}^{\infty}a_k(x-x_0)^k$ *with radius of convergence* R*,*

(i) *the series of real numbers* $\sum\limits_{k=0}^{\infty}a_k(x-x_0)^k$ *converges at each point* x *satisfying* $|x-x_0|<R$.

(ii) $\sum\limits_{k=0}^{\infty}a_k(x-x_0)^k$ *diverges at each* x *satisfying* $|x-x_0|>R$.

PROOF. (i) follows from Theorem 7.5(i) and the fact that an absolutely convergent series is necessarily convergent. (ii) follows from Theorem 7.5(ii). □

EXAMPLE 7.8. *Find the radius of convergence of the power series* $\sum\limits_{n=0}^{\infty}\dfrac{(4x+3)^n}{n^3}$

SOLUTION. Observe that

$$\sum_{n=1}^{\infty}\frac{(4x+3)^n}{n^3}=\sum_{n=1}^{\infty}\frac{4^n}{n^3}\cdot\left(x+\frac{3}{4}\right)^n.$$

Thus

$$R=\frac{1}{\lim\limits_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}}=\frac{1}{\lim\limits_{n\to\infty}\dfrac{4^{n+1}\cdot n^3}{(n+1)^3\cdot 4^n}}=\frac{1}{\lim\limits_{n\to\infty}\dfrac{4}{\left(1+\frac{1}{n}\right)^3}}=\frac{1}{4}.$$

□

8. Interval of convergence.

In view of Corollary 7.7, for a power series $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ with radius of convergence R, the set of points at which $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent form an interval called the *interval of convergence*, which must be either

$$(x_0-R, x_0+R), \qquad (x_0-R, x_0+R],$$
$$[x_0-R, x_0+R) \quad \text{or} \quad [x_0-R, x_0+R].$$

EXAMPLE 8.1. *Find the interval of convergence of the power series.*

$$\textit{(i)} \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2} \qquad \textit{(ii)} \sum_{n=1}^{\infty} \frac{(x-2)^n}{n} \qquad \textit{(iii)} \sum_{n=1}^{\infty} n(x-2)^n$$

SOLUTION. (i). First we find the radius of convergence

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n^2}{(n+1)^2}} = \frac{1}{\lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2}} = 1.$$

Next we check the ending-points $x_0 \pm R = 2 \pm 1 = 1, 3$. When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which is convergent by Example 3.5. When $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is convergent by the p-series. Thus the interval of convergence is $[1, 3]$.

(ii). The radius of convergence is

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n}{(n+1)}} = \frac{1}{\lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})}} = 1.$$

Now we check the ending-points $x_0 \pm R = 1, 3$. When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is convergent by Example 3.5. When $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent by the p-series. Thus the interval of convergence is $[1, 3)$.

(iii). The radius of convergence is

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n+1}{(n)}} = \frac{1}{\lim\limits_{n\to\infty} (1+\frac{1}{n})} = 1.$$

Now we check the ending-points $x_0 \pm R = 1, 3$. When $x = 1$, the series is $\sum_{n=1}^{\infty} n(-1)^n$, and when $x = 3$, the series is $\sum_{n=1}^{\infty} n$. Both of these series are divergent by the divergence test. Thus the interval of convergence is $(1, 3)$. $\square$

THEOREM 8.2. *Suppose that* $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *has radius of convergence* $R>0$ *with pointwise limiting function* $f(x)$ *on* $|x-x_0|<R$ *(i.e.* $f(x)=\sum_{k=0}^{\infty} a_k(x-x_0)^k$ *on* $|x-x_0|<R$*), then* $f(x)$ *has derivatives of all orders on* $|x-x_0|<R$*, and*

$$a_k = \frac{f^{(k)}(x_0)}{k!} \quad \textit{for all } k. \tag{11}$$

(i.e. we have $f(x)=\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$*.)*

PROOF. First we are going to show that the series $\sum_{k=1}^{\infty} ka_k(x-x_0)^{k-1}$ also has radius of convergence R. To see this, we first observe easily that $|ka_k|\ge|a_k|$ for $k\ge1$, and thus

$$\limsup |ka_k|^{\frac{1}{k}} \ge \limsup |a_k|^{\frac{1}{k}}. \tag{12}$$

For any $\epsilon>0$, since $\lim_{k\to\infty} k^{\frac{1}{k}}=1$, it follows that there exists N such that

$$|k^{\frac{1}{k}}-1|<1+\epsilon \quad \text{for all } k>N \Rightarrow \quad k^{\frac{1}{k}}<1+\epsilon \quad \text{for all } k>N$$

$$\Rightarrow \quad |ka_k|^{\frac{1}{k}}<|a_k|^{\frac{1}{k}}(1+\epsilon) \quad \text{for all } k>N \Rightarrow \quad \limsup|ka_k|^{\frac{1}{k}} \le \limsup|a_k|^{\frac{1}{k}}(1+\epsilon).$$

Letting $\epsilon\longrightarrow0$, it follows that

$$\limsup |ka_k|^{\frac{1}{k}} \le \limsup |a_k|^{\frac{1}{k}}.$$

Together with equation 12, it follows that

$$\limsup |ka_k|^{\frac{1}{k}} = \limsup |a_k|^{\frac{1}{k}} = 1/R.$$

Taking reciprocals, it follows that $\sum_{k=1}^{\infty} ka_k(x-x_0)^{k-1}$ also has radius of convergence R.

Next we are going to apply Corollary 6.7 (with $f_k(x)=a_k(x-x_0)^k$) on any interval $|x-x_0|\le\rho$ for any $\rho<R$. Obviously, each $f_k'(x)=ka_k(x-x_0)^{k-1}$ exists and is continuous on $|x-x_0|\le\rho$. Thus condition (i) of Corollary 6.7 is satisfied. Condition (ii) of Corollary 6.7 follows from Theorem 7.5(i). Since the radius of $\sum_{k=1}^{\infty} ka_k(x-x_0)^{k-1}$ is R, by Theorem 7.5 (iii) (applied to $\sum_{k=1}^{\infty} ka_k(x-x_0)^{k-1}$), it follows that $\sum_{k=1}^{\infty} ka_k(x-x_0)^{k-1}$ converges uniformly on $|x-x_0|\le\rho$, i.e. condition (iii) of Corollary 6.7 is also satisfied.

Thus by Corollary 6.7, $f(x)$ is differentiable on $|x-x_0| \leq \rho$, and for all x satisfying $|x-x_0| \leq \rho$, one has

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx} a_k(x-x_0)^k = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1}. \tag{13}$$

In particular, by evaluating at $x = x_0$, one has

$$f'(x_0) = a_1.$$

For any number x satisfying $|x-x_0| < R$, we can choose a number $\rho < R$ such that $|x-x_0| \leq \rho$, and thus $f(x)$ is differentiable at x. Hence $f(x)$ is differentiable and equation 13 holds everywhere on $|x-x_0| < R$.

Repeating the above argument (with $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ replaced by $f'(x) = \sum_{k=1}^{\infty} k a_k(x-x_0)^{k-1}$ on $|x-x_0| < R$), it follows that $f'(x)$ is differentiable on $|x-x_0| < R$, and

$$f''(x_0) = \sum_{k=2}^{\infty} k(k-1) a_k (x-x_0)^{k-2} \Big|_{x=x_0} = 2 \cdot 1 \cdot a_2.$$

Repeating the above argument again and again, it follows that $f(x)$ is has derivatives of all order on $|x-x_0| < R$, and $f^k(x_0) = k! \cdot a_k$ for all k, which leads to equation 11. This finishes the proof of the Theorem. ☐

9. Taylor Series

Definition 9.1. For any function $f(x)$ which has derivatives of all orders at a point $x = x_0$, we may construct the power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,$$

which is called the *Taylor series* of f at $x = x_0$.

Example 9.2. *Find the Taylor series of e^x at $x_0 = 0$.*

Solution. Let $f(x) = e^x$. Then $f^{(n)}(x) = e^x$. Thus $f^{(n)}(0) = 1$ and so the Taylor series of e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots .$$

☐

In view of Theorem 8.2, we may ask the following question:
Question: Does the equality

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \tag{14}$$

hold for $|x-x_0| < R$?
(Here R is the radius of the convergence of the Taylor series.)

It turns out that in general, the answer is NO. (See Example 9.4 for an example of a function such that equation 14 does not hold.

However, the above equality does hold for some elementary functions such as $e^x, \sin x,\ \cos x,\ \ln(1+x)$.

Definition 9.3. Functions for which equation 14 hold are called *analytic functions.*

Example 9.4. Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0, \\ \\ 0 & x = 0. \end{cases}$$

Then we will show that $f(x) \neq$ its Taylor series at $x_0 = 0$.

Proof. First we show that for any $n \in \mathbb{Z}^+$,

$$\lim_{x\to 0} \frac{1}{x^n e^{\frac{1}{x^2}}} = 0. \tag{15}$$

To see this, we substitute $y = \frac{1}{x^2}$ to get

$$\lim_{x\to 0} \frac{1}{x^n e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{1}{x^{2n} e^{\frac{1}{x^2}}} \cdot x^n$$

$$= \left(\lim_{y\to+\infty} \frac{y^n}{e^y}\right) \cdot \lim_{x\to 0} x^n = 0 \cdot 0 \quad \text{(by L'Hopital's rule)} \qquad = 0.$$

Next we compute $f'(0)$.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} - 0}{x} = \lim_{x\to 0} \frac{1}{x e^{\frac{1}{x^2}}} = 0 \quad \text{(by (15))}.$$

For $x \neq 0$,

$$f'(x) = \frac{d}{dx}\big(e^{-\frac{1}{x^2}}\big) = 2x^{-3} e^{-\frac{1}{x^2}}.$$

Thus,

$$f'(x) = \begin{cases} 2x^{-3} e^{-\frac{1}{x^2}} & x \neq 0, \\ \\ 0 & x = 0. \end{cases}$$

Next we compute $f''(x)$.

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{2x^{-3} e^{-\frac{1}{x^2}} - 0}{x} = 2 \lim_{x\to 0} \frac{1}{x^4 e^{\frac{1}{x^2}}} = 0 \quad \text{(by (15))}.$$

Again, for $x \neq 0$,

$$f''(x) = \frac{d}{dx}\big(2x^{-3} e^{-\frac{1}{x^2}}\big) = (-6x^{-4} + 4x^{-6}) e^{-\frac{1}{x^2}}.$$

Similar calculations will lead to

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = f^{(4)}(0) = \cdots = 0.$$

Thus we have

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0 + 0x + 0x^2 + \cdots = 0.$$

Clearly, at any $x \neq 0$, $f(x) = e^{-\frac{1}{x^2}} \neq 0$. Therefore, $f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. $\square$

Some standard analytic functions and their Taylor series.

Some well known analytic functions and their Taylor series at $x = 0$ are given as follows:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (|x| < \infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (|x| < \infty).$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (|x| < \infty).$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (|x| < 1).$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (|x| < 1).$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots \qquad (|x| < 1).$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad (-1 \leq x \leq 1)$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots \qquad (|x| < 1),$$

where

$$\binom{\alpha}{k} = \frac{\alpha \cdot (\alpha - 1) \cdot (\alpha - 2) \cdots (\alpha - k + 1)}{k!}$$

for any real number α and integers $k \geq 1$, and $\binom{\alpha}{0} = 1$

Remark. We may use Theorem 8.2 and the standard Taylor series to find the Taylor series of certain analytic functions.

Example 9.5. *Compute the Taylor series for* $\frac{1}{1+x^2}$ *where* $|x| < 1$ *using the Taylor series for* $\frac{1}{1+x}$ *on the same domain.*

SOLUTION. Since $\dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$, we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n \left(x^2\right)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

for $|x| < 1$. Thus the Taylor series of $\dfrac{1}{1+x^2}$ is $\sum_{n=0}^{\infty}(-1)^n x^{2n}$. □

Let n be a positive integer. The notation $n!!$ means

$$n!! = \begin{cases} 2 \cdot 4 \cdot 6 \cdots n & n = 2k \quad \text{even} \\ \\ 1 \cdot 3 \cdot 5 \cdots n & n = 2k-1 \quad \text{odd} \end{cases}$$

EXAMPLE 9.6. *Find the Taylor series for* $\arcsin x$ *where* $|x| < 1$ *and show that*

$$\frac{\pi}{6} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{k! \cdot (2k+1) \cdot 2^{3k+1}}. \tag{16}$$

SOLUTION. For $|x| < 1$,

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt = \int_0^x \left(1 + (-t^2)\right)^{-\frac{1}{2}} dt = \int_0^x \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-t^2)^k dt$$

$$= \sum_{k=0}^{\infty} \int_0^x \binom{-\frac{1}{2}}{k} (-1)^k t^{2k} dt = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1} = x + \sum_{k=1}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1}$$

Note that

$$\binom{-\frac{1}{2}}{k} = \frac{\left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2} - 1\right) \cdots \left(-\frac{1}{2} - k + 1\right)}{k!} = \frac{\left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdots \left(-\frac{2k-1}{2}\right)}{k!}$$

$$= (-1)^k \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{k! \cdot 2^k} = (-1)^k \frac{(2k-1)!!}{2^k \cdot k!}$$

for $k \geq 1$. Thus

$$\arcsin x = x + \sum_{k=1}^{\infty} (-1)^k (-1)^k \frac{(2k-1)!!}{2^k \cdot k! \cdot (2k+1)} x^{2k+1} = x + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{2^k \cdot k! \cdot (2k+1)} x^{2k+1}$$

and so

$$\frac{\pi}{6} = \arcsin \frac{1}{2} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{2^k \cdot k! \cdot (2k+1) \cdot 2^{2k+1}} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{k! \cdot (2k+1) \cdot 2^{3k+1}}.$$

□

Remark. The remainder of the formula 16 can be estimated as follows.

$$R_n = |S - S_n| = \sum_{k=n+1}^{\infty} \frac{(2k-1)!!}{2^k \cdot k! \cdot (2k+1) \cdot 2^{2k+1}} < \sum_{k=n+1}^{\infty} \frac{1}{(2k+1) 2^{2k+1}}$$

$$
\begin{aligned}
&< \sum_{k=n+1}^{\infty} \frac{1}{(2n+3)2^{2k+1}} = \frac{1}{(2n+3)2^{2n+3}} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right) \\
&= \frac{1}{(2n+3)2^{2n+3}\left(1-\frac{1}{4}\right)} = \frac{1}{3(2n+3)2^{2n+1}}
\end{aligned}
$$

For instance, let $n = 10$, we have

$$\pi \approx 6 \left(\frac{1}{2} + \sum_{k=1}^{10} \frac{(2k-1)!!}{k! \cdot (2k+1) \cdot 2^{3k+1}} \right)$$

with error less than

$$6 \cdot \frac{1}{3 \cdot 23 \cdot 2^{21}} = \frac{1}{23 \cdot 2^{20}} = \frac{1}{24117248}.$$

If we choose $n = 20$, we have

$$\pi \approx 6 \left(\frac{1}{2} + \sum_{k=1}^{20} \frac{(2k-1)!!}{k! \cdot (2k+1) \cdot 2^{3k+1}} \right)$$

with error less than

$$6 \cdot \frac{1}{3 \cdot 43 \cdot 2^{41}} = \frac{1}{43 \cdot 2^{40}} = \frac{1}{47278999994368} < 10^{-13}.$$

If $n = 40$, the error is less than $\dfrac{1}{100340843028014221500612608} < 10^{-26}$.

10. Taylor Formula and its Applications

THEOREM 10.1. *If f and its first $(n+1)$ derivatives f', f'', $\cdots$, $f^{(n+1)}$ are continuous on $[x_0, x]$ or $[x, x_0]$, then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x),$$

where $R_n(x) = \dfrac{1}{n!} \displaystyle\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt.$

PROOF. We may assume that $x_0 < x$. Let $g(t) = f(x) - \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$ be a function on t for $x_0 \le t \le x$. Then

$$
\begin{aligned}
g'(t) &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!} k (x-t)^{k-1} \cdot (-1) \\
&\quad -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \\
&= -\left[\frac{f'(t)}{0!} + \frac{f''(t)}{1!}(x-t) + \cdots + \frac{f^{(n+1)}(t)}{n!}(x-t)^n \right] \\
&\quad + \left[\frac{f'(t)}{0!} + \frac{f''(t)}{1!}(x-t) + \cdots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} \right]
\end{aligned}
$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Thus $g(x) - g(x_0) = \displaystyle\int_{x_0}^{x} g'(t)dt = -\int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = -R_n(x)$ and so

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = g(x_0) = R_n(x)$$

because $g(x) = 0$. □

LEMMA 10.2. *Let f and g be continuous on $[a, b]$. Suppose that g does not change sign in $[a, b]$. Then there is a point $\xi \in (a, b)$ such that*

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx.$$

PROOF. We may assume that $g(x) \geq 0$ for $x \in [a, b]$. Since f is continuous on $[a, b]$, f has maximum and minimum on $[a, b]$. Let $M = \max\{f(x)|a \leq x \leq b\}$ and $m = \min\{f(x)|a \leq x \leq b\}$. Then $m \leq f(x) \leq M$ for $a \leq x \leq b$ and so

$$m\int_a^b g(x)dx = \int_a^b mg(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$$

If $\displaystyle\int_a^b g(x)dx = 0$, then $\displaystyle\int_a^n f(x)g(x)dx = 0$ and the assertion holds in this case. Otherwise $\displaystyle\int_a^b g(x)dx > 0$ and so

$$m \leq \frac{\displaystyle\int_a^b f(x)g(x)dx}{\displaystyle\int_a^b g(x)dx} \leq M.$$

By the intermediate-value theorem, there is a point ξ between a and b such that

$$f(\xi) = \frac{\displaystyle\int_a^b f(x)g(x)dx}{\displaystyle\int_a^b g(x)dx}.$$

□

THEOREM 10.3 (Taylor Formula.). *If f has derivatives of all orders in an open interval I containing x_0, then for each positive integer n and for each x in I,*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x),$$

where the remainder

$$R_n(x) = \frac{f^{(n+1}(\xi))}{(n+1)!}(x-x_0)^{n+1}.$$

PROOF. Since $g(t) = (x-t)^n$ does not change the sign on $[x_0, x]$, by the above lemma, we have

$$\begin{aligned} R_n(x) &= \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt = \frac{f^{(n+1)}(\xi)}{n!}\int_{x_0}^{x}(x-t)^n dt \\ &= \frac{f^{(n+1)}(\xi)}{n!}\frac{(x-x_0)^{n+1}}{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

□

Note. The Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ converges to $f(x)$, that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,$$

if and only if the Taylor remainder $R_n(x) \to 0$ as $n \to \infty$. The function in Example 9.4 is not analytic because the Taylor remainder does not tends to 0 for any $x \neq 0$.

Taylor Estimation. If f has derivatives of all orders in an open interval I containing x_0, then for each positive integer n and for each x in I,

$$f(x) \approx T_n(f) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

with error

$$|R_n(x)| \le \frac{\max\limits_{t\in[x_0,x]}\{f^{(n+1}(t))\}}{(n+1)!}|x-x_0|^{n+1}$$

EXAMPLE 10.4. *Show that* $e^x = \sum_{k=0}^{\infty} \frac{x^k}{n!}$ *for* $x \in (-\infty, +\infty)$.

SOLUTION. Let $f(x) = e^x$ and $x_0 = 0$. Then $f^{(k)}(x) = e^x$ and so $f^{(k)}(0) = 1$ for all $k \ge 0$. Since the Taylor remainder

$$|R_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\right| \le \left|\frac{\max\{e^x, 1\}\cdot x^{n+1}}{(n+1)!}\right| \to 0$$

as $n \to \infty$ for any given x. Thus e^x equals to its Taylor series for any given x, that is,

$$e^x = \sum_{k=0}^{\infty} \frac{1}{n!}x^k = 1 + x + \frac{x^2}{2!} + \cdots$$

□

Applications

Binomial Series. Let m be any constant. Recall that the binomial series states that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

for $|x| < 1$.

EXAMPLE 10.5.

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \cdots$$

$$\sqrt{1-x^3} = (1-x^3)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(-x^3\right)^k = 1 - \frac{1}{2}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^6 + \cdots$$

$$\sqrt{4.1} = \left(4 + \frac{1}{10}\right)^{\frac{1}{2}} = 2\left(1 + \frac{1}{40}\right)^{\frac{1}{2}} = 2\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{1}{40}\right)^k.$$

Evaluating Integrals.

EXAMPLE 10.6. *Evaluate the integral* $\int_0^1 \sin x^2 dx$ *with an error of less than to 0.001.*

SOLUTION. From $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, we have

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{4k-2}}{(2k-1)!}$$

and so

$$\int_0^1 \sin x^2 dx = \int_0^1 x^2 dx - \int_0^1 \frac{x^6}{3!} dx + \int_0^1 \frac{x^{10}}{5!} dx + \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \int_0^1 \frac{x^{4k-2}}{(2k-1)!} dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)!(4k-1)}$$

is an alternating series. Let $a_n = \frac{1}{(2n-1)!(4n-1)}$. From

$$a_{n+1} = \frac{1}{(2n+1)!(4n+3)} < \frac{1}{1000},$$

we have $(2n+1)!(4n+3) > 1000$ or $n \geq 3$. Thus

$$\int_0^1 \sin x^2 dx \approx \frac{1}{1! \cdot 3} - \frac{1}{3! \cdot 7} + \frac{1}{5! \cdot 11}$$

with error less that 0.001. □

EXAMPLE 10.7. For $|x| < 1$, we have the Taylor series

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k} dt$$

$$= \sum_{k=0}^{\infty}(-1)^k \int_0^x t^{2k}dt = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$$

for $|x| < 1$. This formula also holds for the ending points $x = \pm 1$ (We omit the proof of this!), that is,

$$\arctan x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$$

for $|x| \leq 1$ and so

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots .$$

Evaluating Limits.

Example 10.8.

$$\lim_{x\to 0} \frac{\sin(x^2) - \arctan(x^2)}{x^6}$$

$$= \lim_{x\to 0} \frac{\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right) - \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \cdots\right)}{x^6}$$

$$= \lim_{x\to 0} \left(-\frac{1}{3!} + \frac{1}{3}\right) + \left(\frac{1}{5!} - \frac{1}{5}\right) x^4 + \cdots = -\frac{1}{3!} + \frac{1}{3} = \frac{1}{6}.$$

Example 10.9. *Let* $f(x) = \sqrt[3]{1 + x^2}$. *Find* $f^{(10)}(0)$.

Solution. From

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(x) = (1 + x^2)^{\frac{1}{3}} = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} \left(x^2\right)^k$$

$$= \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} x^{2k}$$

for $|x| < 1$, we have $\frac{f^{(10)}(0)}{10!} = \binom{\frac{1}{3}}{5}$ by comparing the coefficients of x^{10} and so

$$f^{(10)}(0) = 10!\cdot\binom{\frac{1}{3}}{5} = 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot \frac{1}{3}\cdot\left(\frac{1}{3} - 1\right)\cdot\left(\frac{1}{3} - 2\right)\cdot\left(\frac{1}{3} - 3\right)\cdot\left(\frac{1}{3} - 4\right) = \frac{985600}{9}.$$

□

CHAPTER 4

Ordinary Differential Equations (ODE)

1. Classification of Differential Equations

A differential equation (DE) is an equation involving an unknown function and its derivatives.

Example 1.1. $4\frac{d^2y}{dt^2}+\frac{dy}{dt}+2y=0$ is a DE with y being the function (or dependent variable) and t being the independent variable.

Remark. The word 'ordinary' in the heading of this chapter means that the differential equation involves only 'ordinary' derivatives of the function (rather than partial derivatives of the function). If you had taken MA1104, then you should know that the derivatives of a function in two or more variables, say $f(x, y)$, are known as partial derivatives. In this course, we deal with functions of one variable only, thus the word 'ordinary' carries no special meaning to us.

Classifying DE by its order

The *order* of a differential equation (DE) is the order of its highest derivative.

Example 1.2. Determine the order of the following.

1. $4\frac{d^2y}{dt^2}+\frac{dy}{dt}+2y=0$ is a 2nd order DE.
2. $\frac{dy}{dt}-2ty=t$ is a 1st order DE
3. $(\frac{dy}{dt})^3-2ty^3=0$ is a 1st order ODE.

Linear or non-linear DE

The *total power* of a term in a DE is the sum of all the powers of y and its derivatives in that term.

A *linear* DE is a DE where the total power of each term is at most 1.

Example 1.3. Discuss following differential equations.

1. $4\frac{d^2y}{dt^2}+\frac{dy}{dt}+2y=0$

2. $\frac{dy}{dt}-2ty=t$:

3. $y\frac{dy}{dt}-2ty=t$

Homogeneous or non-homogeneous DE

A *homogeneous* DE is a DE where the total power of each non-zero term is the same.

EXAMPLE 1.4. Discuss the following equations

1. $4\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}+2y=0.$

2. $\dfrac{dy}{dt}-2ty=t.$

3. $y\dfrac{dy}{dt}-2ty^2=0$

We may combine the classifications:

EXAMPLE 1.5. Classify the following equations

1. $4\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}+2y=0$

   is a homogeneous/non-homogeneous, linear/non-linear,   order ODE.

2. $\dfrac{dy}{dt}-2ty=t$

3. $\left(\dfrac{dy}{dt}\right)^3-2ty=t$

2. General solutions of homogeneous linear first order ODE

This is the easiest DE to solve:

$$\frac{dy}{dt}+g(t)y=0. \tag{17}$$

Separation of variables.

The method is to "separate the variables", y and t. Equation 17 can be rearranged as

$$\frac{dy}{y}=-g(t)dt.$$

Upon integrating both sides, we get

$$\int\frac{dy}{y}=-\int g(t)dt \ln|y|=-\int g(t)dt+k,$$

$$|y|=e^k\exp\Big(-\int g(t)dt\Big).$$

The solution is then

$$y(t)=C\exp\Big(-\int g(t)dt\Big), \tag{18}$$

where C is an arbitrary constant. The above solution is said to be the *general solution* of the DE, equation 17, since every solution of equation 1 must be of this form.

EXAMPLE 2.1. *Find the general solution of* $\frac{dy}{dt} - 2ty = 0$.

SOLUTION. Since $g(t) = -2t$,

$$y(t) = Ce^{-\displaystyle\int (-2t)\,dt} = Ce^{t^2}.$$

$\square$

3. General solutions of non-homogeneous linear first order ODE

The DE to be considered is:

$$\frac{dy}{dt} + g(t)y = f(t). \tag{19}$$

There are essentially 2 methods to solve this:
Method 1: Multiply equation 19 by an Integrating Factor, $\mu(t)$, where

$$\mu(t) = exp\left(\int g(t)dt\right). \tag{20}$$

We get

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \mu(t)f(t).$$

Using equation 20, it is easy to check that

$$\frac{d}{dt}\Big(\mu(t)y\Big) = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y,$$

so we get

$$\frac{d}{dt}\Big(\mu(t)y\Big) = \mu(t)f(t)$$

$$\Rightarrow \qquad \mu(t)y = \int \mu(t)f(t)dt + C. \tag{21}$$

Remark. This method is not generally applicable to higher order linear DE.

EXAMPLE 3.1. *Find the general solution of* $\frac{dy}{dt} - 2ty = t$.

SOLUTION. We have

$$\mu(t) = e^{\displaystyle\int g(t)\,dt} = e^{\displaystyle\int -2t\,dt} = e^{-t^2}$$

and

$$\begin{aligned} y = \frac{1}{\mu(t)}\left(\int \mu(t)f(t)dt + C\right) &= e^{t^2}\left(\int e^{-t^2}\cdot t\,dt + C\right) \\ e^{t^2}\left(-\frac{1}{2}\int e^{-t^2}\,d(-t^2) + C\right) &= e^{t^2}\left(-\frac{1}{2}e^{-t^2} + C\right) \\ &= Ce^{t^2} - \frac{1}{2}. \end{aligned}$$

$\square$

Method 2: For a linear DE, say, equation 19, the general solution can be written as

$$y(t) = y_h(t) + y_p(t), \tag{22}$$

where y_h is the general solution of the corresponding homogeneous DE

$$\frac{dy}{dt} + g(t)y = 0,$$

while y_p is any one solution (called a *particular* solution) of equation 19.

EXAMPLE 3.2. *Find the general solution of* $\frac{dy}{dt} - 2ty = t$.

PROOF. Clearly $y_p = -\frac{1}{2}$ is a particular solution. By Example 2.1, $y_h = Ce^{t^2}$ and so

$$y(t) = y_h + y_p = Ce^{t^2} - \frac{1}{2}.$$

□

4. Some non-linear first order ODEs

We remark that the method of separation of variables can also be used to solve certain non-homogeneous non-linear first order ODE of the form

$$\frac{dy}{dt} + f(t)g(y) = 0, \quad \text{or} \quad \frac{dy}{dt} = f(\frac{y}{t}). \tag{23}$$

For the first equation, we have

$$\frac{dy}{g(y)} = -f(t)dt \qquad \int \frac{1}{g(y)}\,dy = -\int f(t)dt + C. \tag{24}$$

For the second equation, let $v = \frac{y}{t}$, then $y = vt$ and $y' = v't + v$. Thus we have

$$tv' + v = f(v) \quad \Rightarrow \quad tv' = f(v) - v \quad \Rightarrow \quad \int \frac{dv}{f(v) - v} = \int \frac{dt}{t} + C \tag{25}$$

EXAMPLE 4.1. *Find the general solution of* $\frac{dy}{dt} = \frac{t}{y^2}$.

SOLUTION.

$$\int y^2\,dy = \int t\,dt \quad \Rightarrow \quad \frac{y^3}{3} = \frac{t^2}{2} + C \quad \Rightarrow \quad y = \sqrt[3]{\frac{3t^2}{2} + 3C}.$$

□

5. Homogeneous linear 2nd order ODE with constant coefficients

Homogeneous linear 2nd order ODE with constant coefficients are of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{26}$$

where a, b, c are real constants.

Example 5.1. Below are some examples

1. $2\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - 5y = 0.$
2. $y'' + 5y' + 4y = 0.$

Remark: The general solution of equation 26 is of the form

$$y(t) = Ay_1(t) + By_2(t), \tag{27}$$

where A, B are arbitrary constants, and $y_1(t)$, $y_2(t)$ are solutions of equation 26 such that $y_1(t)$ and $y_2(t)$ are not constant multiples of each other (see §2.1, p. 129-130 of [3]).

To solve equation 26, we look at the case of the homogeneous linear 1st order ODE with constant coefficients, which is of the form

$$y' + ky = 0, \qquad k \in \mathbb{R}.$$

Its general solution is of the form $y(t) = Ae^{-kt}$.

Thus, to solve equation 26, we try

$$y(t) = e^{rt}, \quad \text{where } r \in \mathbb{R}. \tag{28}$$

Then we have

$$y'(t) = re^{rt}, \quad y''(t) = r^2e^{rt}. \tag{29}$$

Substituting equations 28 and 29 into 26, equation 26 becomes

$$ay'' + by' + cy = 0 \quad \Rightarrow \quad ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$\Rightarrow \quad e^{rt}(ar^2 + br + c) = 0 \quad \Rightarrow \quad ar^2 + br + c = 0.$$

The quadratic equation

$$ar^2 + br + c = 0 \tag{30}$$

is known as the *auxiliary* (or *characteristic*) equation of the homogeneous linear second order ODE, equation 26. We denote its two roots by r_1, r_2. The general solution $y(t)$ for equation 26 will depend on what kind of roots r_1 and r_2 are.

Case (a): r_1, r_2 are real and distinct ($r_1 \neq r_2$).

In this case, the general solution of equation 26 is

$$y(t) = Ae^{r_1t} + Be^{r_2t}. \tag{31}$$

By an earlier remark, it suffices to check that $y_1(t) = e^{r_1t}$ and $y_2(t) = e^{r_2t}$ are not constant multiples of each other and that $y_1(t)$ and $y_2(t)$ are indeed solutions of equation 26.

Check: $\dfrac{y_1(t)}{y_2(t)} = e^{(r_1-r_2)t}$ is not a constant since $r_1 \neq r_2$. Hence $y_1(t)$ and $y_2(t)$ are not constant multiples of each other. Also, substituting $y_1(t)$ into equation 26, we get

$$LHS = ay_1''(t)+by_1'(t)+cy_1(t) = ar_1^2e^{r_1t}+br_1e^{r_1t}+ce^{r_1t} = e^{r_1t}(ar_1^2+br_1+c) = 0 = RHS.$$

Hence $y_1(t)$ is indeed a solution of equation 26. Similarly, $y_2(t)$ is also a solution of equation 26.

Case (b): r_1, r_2 are (non-real) complex numbers. Write $r_1, r_2 = \alpha \pm i\beta$, where α, $\beta \in \mathbb{R}$ with $\beta \neq 0$.

In this case, the general solution of equation 26 is

$$y(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t. \tag{32}$$

Again, it suffices to check that $y_1(t) = e^{\alpha t}\cos\beta t$ and $y_2(t) = e^{\alpha t}\sin\beta t$ are not constant multiples of each other, and $y_1(t)$ and $y_2(t)$ are solutions of equation 26.

Check: $\dfrac{y_2(t)}{y_1(t)} = \tan\beta t$ is not a constant since $\beta \neq 0$. Hence $y_1(t)$ and $y_2(t)$ are not constant multiples of each other. Since $\alpha \pm i\beta$ are roots of the characteristic equation, we have

$$\text{Sum of roots} = 2\alpha = -\frac{b}{a} \tag{33}$$

$$\text{Product of roots} = \alpha^2 + \beta^2 = \frac{c}{a} \tag{34}$$

Differentiating $y_1(t)$, we get

$$y_1'(t) = \alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t}\sin\beta t,$$

$$y_1''(t) = \alpha^2 e^{\alpha t}\cos\beta t - 2\alpha\beta e^{\alpha t}\sin\beta t - \beta^2 e^{\alpha t}\cos\beta t.$$

Substituting $y_1(t)$ into equation 26, we get

$$\begin{aligned}
LHS = ay_1''(t) + by_1'(t) + cy_1(t) &= a[\alpha^2 e^{\alpha t}\cos\beta t - 2\alpha\beta e^{\alpha t}\sin\beta t - \beta^2 e^{\alpha t}\cos\beta t] \\
&\quad + b[\alpha e^{\alpha t}\cos\beta t - \beta e^{\alpha t}\sin\beta t] + ce^{\alpha t}\cos\beta t \\
&= [a(\alpha^2 - \beta^2) + b\alpha + c]e^{\alpha t}\cos\beta t - (2a\alpha + b)\beta e^{\alpha t}\sin\beta t \\
&= [a(\alpha^2 - \beta^2) + (-2a\alpha)\alpha + c]e^{\alpha t}\cos\beta t - 0 \quad \text{(by (33))} \\
&= [-a(\alpha^2 + \beta^2) + c]e^{\alpha t}\cos\beta t = 0 \quad \text{(by (34))} = RHS.
\end{aligned}$$

Hence $y_1(t)$ is indeed a solution of equation 26. Similarly, $y_2(t)$ is also a solution of equation 26.

Case (c): $r_1 = r_2 (= r)$.

In this case, the general solution of equation 26 is

$$y(t) = (At + B)e^{rt}. \tag{35}$$

Again it suffices to check that $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ are not constant multiples of each other, and they are solutions of equation 26.

Check: $\dfrac{y_2(t)}{y_1(t)} = t$ is not a constant. Hence $y_1(t)$ and $y_2(t)$ are not constant multiples of each other. Also, one can show as in case (a) that $y_1(t)$ is a solution of equation 26. To check that $y_2(t)$ is a solution of equation 26, we first observe that

$$\text{Sum of roots} = 2r = -\frac{b}{a}. \tag{36}$$

Differentiating $y_2(t)$, we get

$$y_2'(t) = rte^{rt} + e^{rt}, \quad y_2''(t) = r^2 te^{rt} + 2re^{rt}.$$

Substituting into equation 26, we get

$$\begin{aligned} LHS &= ay_2''(t) + by_2'(t) + cy_2(t) = a[r^2te^{rt} + 2re^{rt}] + b[rte^{rt} + e^{rt}] + cte^{rt} \\ &= (ar^2 + br + c)te^{rt} + (2ar + b)e^{rt} = 0 + 0 \quad \text{(by (36))} = RHS. \end{aligned}$$

Hence $y_2(t)$ is indeed a solution of equation 26.

6. Non-homogeneous linear 2nd order ODE with constant coefficients

Non-homogeneous linear 2nd order ODE with constant coefficients are of the general form

$$ay'' + by' + cy = g(t), \tag{37}$$

where $a, b, c \in \mathbb{R}$, and $g(t)$ is a function in t.

EXAMPLE 6.1. $y'' + y = t^2$.

Associated with the non-homogeneous equation 37 is a homogeneous equation given by

$$ay'' + by' + cy = 0. \tag{38}$$

THEOREM 6.2. *The general solution of equation 37 is given by*

$$y(t) = y_h(t) + y_p(t), \tag{39}$$

where $y_h(t)$ is the general solution of the associated homogeneous equation 38, and $y_p(t)$ is a particular solution of equation 37.

PROOF. Let $y(t)$ be any solution of equation 37, and let $y_p(t)$ be a particular solution of equation 37. Let $f(t) = y(t) - y_p(t)$. Then

$$\begin{aligned} af'' + bf' + cf &= a(y''(t) - y_p''(t)) + b(y'(t) - y_p'(t)) + c(y(t) - y_p(t)) \\ &= (ay''(t) + by'(t) + cy(t)) - (ay_p''(t) + by_p'(t) + cy_p(t)) = g(t) - g(t) = 0. \end{aligned}$$

Thus $f(t)$ satisfies equation 38.

Similarly, we can show that any function of the form $y_h(t) + y_p(t)$ satisfies equation 37.

Thus the general solution of equation 37 is of the form $y(t) = y_h(t) + y_p(t)$ and hence the result. ☐

So the problem of solving the non-homogeneous 2nd order linear ODE with constant coefficients is reduced to finding $y_h(t)$ and $y_p(t)$.

$y_h(t)$ can be found by the method explained in section 5 of this chapter.

Finding $y_p(t)$ amounts to guesswork. But for certain kind of $g(t)$ (the non-homogeneous term of equation 37), the guesswork is quite systematic. This is explained below.

Judicious guessing/the method of undetermined coefficients

We want to find one particular solution $y_p(t)$ for the non-homogeneous equation

$$ay'' + by' + cy = g(t). \tag{40}$$

The homogeneous equation associated to equation 40 is

$$ay'' + by' + cy = 0, \tag{41}$$

with characteristic equation given by

$$ar^2 + br + c = 0. \tag{42}$$

Suppose r_1, r_2 are the roots of the characteristic equation 42. We look for particular solution $y_p(t)$ of equation 40 based on the form of $g(t)$.

Case (a): $g(t) = a_0 + a_1 t + \cdots + a_n t^n$.

Then $y_p(t)$ is of the form

$$y_p(t) = \begin{cases} A_0 + A_1 t + \cdots + A_n t^n, & c \neq 0 \\ t(A_0 + A_1 t + \cdots + A_n t^n), & c = 0,\ b \neq 0 \\ t^2(A_0 + A_1 t + \cdots + A_n t^n), & c = b = 0. \end{cases}$$

Case (b): $g(t) = (a_0 + a_1 t + \cdots + a_n t^n)e^{\alpha t}$.

The form of $y_p(t)$ will depend on whether α is a root of the characteristic equation. Try

$$y_p(t) = \begin{cases} (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t} & r_1, r_2 \neq \alpha \quad \text{i.e. } \alpha \text{ is not a root)} \\ t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t} & r_1 = \alpha,\ r_2 \neq \alpha \quad \text{i.e. } \alpha \text{ is a single root)} \\ t^2(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t} & r_1, r_2 = \alpha \quad (\alpha \text{ is a double root)} \end{cases}$$

Case (c):

$$g(t) = (a_0 + a_1 t + \cdots + a_n t^n)e^{\alpha t}\cos\beta t, \quad \text{or}$$
$$g(t) = (a_0 + a_1 t + \cdots + a_n t^n)e^{\alpha t}\sin\beta t.$$

The form of $y_p(t)$ will depend on whether $\alpha \pm i\beta$ are roots of the characteristic equation or not.

(i) If $r_1, r_2 \neq \alpha \pm i\beta$ (i.e. $\alpha \pm i\beta$ are not roots), try

$$y_p(t) = (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}\cos\beta t + (B_0 + B_1 t + \cdots + B_n t^n)e^{\alpha t}\sin\beta t.$$

(ii) If $r_1, r_2 = \alpha \pm i\beta$ (i.e. $\alpha \pm i\beta$ are roots), try

$$y_p(t) = t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}\cos\beta t + t(B_0 + B_1 t + \cdots + B_n t^n)e^{\alpha t}\sin\beta t.$$

Remark. 1. Once you have determined the form of y_p, substitute into equation 40 to get equations in the unknown coefficients $A_0, \cdots, A_n$ and also $B_0, \cdots, B_n$ if necessary. It should be easy to solve for these coefficients.

2. Since the DE is linear, if $g(t)$ are sums of the above three types, we can still find y_p by adding the corresponding sums of the individual y_p.

Supplement: A General Formula

We are going to find a formula for determining a particular solutions $y_p(t)$ of the differential equation

$$ay'' + by' + cy = g(t) \tag{1}$$

with $a \neq 0$. Recall that the associated homogeneous equation is

$$ay'' + by' + cy = 0. \tag{2}$$

Let $y_h(t) = Ay_1(t) + By_2(t)$ be the general solution of the equation (2). We try

$$y_p(t) = A(t)y_1(t) + B(t)y_2(t) \tag{3}$$

for *undetermined* functions $A(t)$ and $B(t)$ satisfying

$$A'(t)y_1(t) + B'(t)y_2(t) = 0 \qquad (4).$$

Note. When we plug y_p into equation (1), we obtain an equation for the functions $A(t)$ and $B(t)$. Technically we make equation (4) as an additional equation such that we will be able to find $A(t)$ and $B(t)$.

Now we compute y_p' and y_p''.

$$y_p' = (A'y_1 + B'y_2) + (Ay_1' + By_2') = Ay_1' + By_2' \tag{5}$$

$$y_p'' = A'y_1' + B'y_2' + Ay_1'' + By_2'' \tag{6}$$

From equation (1), we have

$$g(t) = a(A'y_1' + B'y_2' + Ay_1'' + By_2'') + b(Ay_1' + By_2') + c(Ay_1 + By_2)$$

$$= aA'y_1' + aB'y_2' + A(ay_1'' + by_1' + cy_1') + B(ay_2'' + by_2' + cy_2) = aA'y_1' + aB'y_2'.$$

Thus we have the equation

$$\begin{cases} aA'(t)y_1'(t) + aB'(t)y_2'(t) = g(t) \\[2ex] A'(t)y_1(t) + B'(t)y_2(t) = 0 \end{cases}$$

It follows that

$$\begin{cases} A'(t) = \dfrac{\begin{vmatrix} g(t) & ay_2'(t) \\ 0 & y_2(t) \end{vmatrix}}{\begin{vmatrix} ay_1'(t) & ay_2'(t) \\ y_1(t) & y_2(t) \end{vmatrix}} = \dfrac{y_2(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)} \\[6ex] B'(t) = \dfrac{\begin{vmatrix} ay_1'(t) & g(t) \\ y_1(t) & 0 \end{vmatrix}}{\begin{vmatrix} ay_1'(t) & ay_2'(t) \\ y_1(t) & y_2(t) \end{vmatrix}} = \dfrac{-y_1(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)} \end{cases}$$

and so we have the formula

$$y_p(t) = \left(\int \frac{g(t)y_2(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right)\cdot y_1(t) + \left(\int \frac{-y_1(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right)\cdot y_2(t)$$

Example 1. Solve $y'' + y = t^2$.

Solution. From $y'' + y = 0$, we have $r^2 + 1 = 0$ or $r = \pm i$.

$$y_h = A\cos t + B\sin t,$$

where $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

$$\Delta = \begin{vmatrix} ay_1'(t) & ay_2'(t) \\ y_1(t) & y_2(t) \end{vmatrix} = \begin{vmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1.$$

$$\begin{aligned} y_p(t) &= \left(\int \frac{g(t)y_2(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right)\cdot y_1(t) + \left(\int \frac{-y_1(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right)\cdot y_2(t) \\ &= \left(\int \frac{t^2 \sin t}{-1}dt\right)\cos t + \left(\int \frac{-t^2\cos t}{-1}dt\right)\sin t \end{aligned}$$

Observe that

$$\begin{aligned} \int -t^2 \sin t dt &= \int t^2 d\cos t = t^2\cos t - \int 2t\cos t dt = t^2 \cos t - 2\int t d\sin t \\ &= t^2\cos t - 2t\sin t + 2\int \sin t dt = t^2\cos t - 2t\sin t - 2\cos t \\ \int t^2 \cos t dt &= \int t^2 d\sin t = t^2 \sin t - \int 2t\sin t dt = t^2\sin t + 2\int t d\cos t \\ &= t^2 \sin t + 2t\cos t - 2\int \cos t dt = t^2\sin t + 2t\cos t - 2\sin t \end{aligned}$$

Thus

$$y_p(t) = (t^2\cos t - 2t\sin t - 2\cos t)\cos t + (t^2\sin t + 2t\cos t - 2\sin t)\sin t = t^2 - 2$$

and so

$$y(t) = A\cos t + B\sin t + t^2 - 2.$$

□

Example 2. Solve $y'' + y = \dfrac{1}{\cos t}$.

Solution. From $y'' + y = 0$, we have $r^2 + 1 = 0$, $r = \pm i$ and so $y_h = A\cos t + B\sin t$, where $y_1(t) = \cos t$ and $y_2(t) = \sin t$. As above,

$$\Delta = \begin{vmatrix} ay_1'(t) & ay_2'(t) \\ y_1(t) & y_2(t) \end{vmatrix} = \begin{vmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1.$$

and so

$$\begin{aligned} y_p(t) &= \left(\int \frac{g(t)y_2(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right)\cdot y_1(t) + \left(\int \frac{-y_1(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right)\cdot y_2(t) \\ &= \left(\int -\frac{\sin t}{\cos t}dt\right)\cos t + \left(\int \frac{\cos t}{\cos t}dt\right)\sin t = (\ln|\cos t|)\cos t + t\sin t \end{aligned}$$

and so

$$y(t) = A\cos t + B\sin t + (\ln|\cos t|)\cos t + t\sin t$$

□