

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2001-2002

**MA2108    Advanced Calculus II**

November 2001 — Time allowed : 2 hours

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**INSTRUCTIONS TO CANDIDATES**

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

**SECTION A**

Answer **all** the questions in this section. Section A carries a total of 60 marks.

**Question 1** [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

- (a)  $\left\{ \sqrt[3]{\frac{8^n + n^8 + n!}{n! + (\ln n)^8}} \right\}$ .
- (b)  $\left\{ \frac{\sqrt{n}}{\sqrt{2n+1} - \sqrt{n}} \right\}$ .
- (c)  $\left\{ \left( \frac{2n}{2n-1} \right)^{3n} \right\}$ .
- (d)  $\left\{ 3n \sin \left( \frac{1}{n} \right) + (5^n + 3^n)^{\frac{1}{n}} \right\}$ .

**Question 2** [16 marks]

Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

- (a)  $\sum_{n=1}^{\infty} \frac{1}{n + \ln n}$ .
- (b)  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2\sqrt{n+1}}$ .
- (c)  $\sum_{n=1}^{\infty} (-1)^n \frac{1 + \sin n}{n(1 + \ln n)^2}$ .
- (d)  $\sum_{n=1}^{\infty} \frac{2^{4n} (n!)^3}{(3n)!}$ .

**Question 3** [12 marks]

Find the interval of convergence of the following power series.

$$(a) \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n^2 - \ln n}.$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{(2x - 1)^n}{\sqrt{n}}.$$

**Question 4** [16 marks]

(a) Find the general solutions of the following differential equations.

$$(i) \quad y' - x^2y = y.$$

$$(ii) \quad y'' + 5y' + 6y = 0.$$

(b) Evaluate the limit

$$\lim_{t \rightarrow 0} \frac{1 - \cos(t^5)}{t \sin(t^3) - t^4}.$$

**SECTION B**

Answer not more than **TWO (2)** questions from this section. Each question in this section carries 20 marks.

**Question 5** [20 marks]

(a) Evaluate  $\lim_{n \rightarrow \infty} \int_0^1 \left( \frac{x^2 + x + 1}{4} \right)^n \sin nx dx$ . Justify your answer.

(b) Find the first three terms (up to and including the quadratic term) for the Taylor series for the function  $f(x) = \sqrt[3]{x}$  expanded around  $x = 8$ .

(c) Use an appropriate series to find a decimal approximation to  $\sqrt[3]{8.1}$  that is accurate to within  $10^{-6}$ . Justify your answer.

**Question 6** [20 marks]

- (a) Solve the following initial value problem:

$$y'' + 2y' + y = te^{-2t}; \quad y(0) = 0 \quad y'(0) = 2.$$

- (b) Consider the sequence
- $\{a_n\}$
- given by

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}, \quad \text{for } n \geq 1.$$

Show that  $\{a_n\}$  converges, and find its limit.

- (c) Does the series of functions
- $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x}$
- converge uniformly on the interval
- $(0, +\infty)$
- ? Justify your answer.

**Question 7** [20 marks]

- (a) Let
- $\{a_n\}$
- be a bounded sequence of real numbers. Show that

$$\limsup_{n \rightarrow \infty} \sqrt{|a_n|} = \sqrt{\limsup_{n \rightarrow \infty} |a_n|}.$$

- (b) (i) Let
- $a > 1$
- be any positive number greater than 1. Show that the series of functions
- $\sum_{n=1}^{\infty} \frac{\ln n}{n^x}$
- converges uniformly over the interval
- $[a, +\infty)$
- .

- (ii) Using (i) or otherwise, show that the function
- $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$
- is differentiable on
- $(1, +\infty)$
- .