Supplement to Section 3.9

Binomial Series. Let m be any real constant. Show that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n$$

for $|x| < 1$, where

$$\binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$$

for any real number m and positive integers n.

Proof. First we show that the power series

$$f_m(x) = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n$$

converges pointwise on $|x| < 1$. Let $a_n = \binom{m}{n} x^n$. Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|m| \cdot |m-1| \cdots |m-n| \cdot n!}{(n+1)! \cdot |m| \cdot |m-1| \cdots |m-n+1|}$$

$$= \lim_{n\to\infty} \frac{|m-n|}{(n+1)} = \lim_{n\to\infty} \frac{|m/n - 1|}{1 + 1/n} = 1.$$

Thus the radius of convergence

$$R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = 1$$

and so the series

$$\sum_{n=0}^{\infty} \binom{m}{n} x^n$$

is absolutely convergent by the ratio test for $|x| < 1$ and so it converges for $|x| < 1$.

Next we are going to set up a differential equation. By Theorem **??**, we have

$$f'_m(x) = \sum_{n=1}^{\infty} \binom{m}{n} n x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{m(m-1)\cdots(m-n+1)\cdot n}{n!} x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^{n-1}$$

$$= m\sum_{n=1}^{\infty}\binom{m-1}{n-1}x^{n-1}.$$

Now

$$\begin{aligned}(1+x)f_m'(x) &= f_m'(x) + xf_m'(x)\\ &= m\sum_{n=1}^{\infty}\binom{m-1}{n-1}x^{n-1} + m\sum_{n=1}^{\infty}\binom{m-1}{n-1}x^{n}\\ &= m\sum_{n=0}^{\infty}\binom{m-1}{n}x^{n} + m\sum_{n=1}^{\infty}\binom{m-1}{n-1}x^{n}\\ &= m\left\{1+\sum_{n=1}^{\infty}\left[\binom{m-1}{n}+\binom{m-1}{n-1}\right]x^n\right\}\\ &= m\left[1+\sum_{n=1}^{\infty}\binom{m}{n}x^n\right] = mf_m(x),\end{aligned}$$

where

$$\begin{aligned}&\binom{m-1}{n}+\binom{m-1}{n-1}\\ &= \frac{(m-1)(m-2)\cdots(m-1-n+1)}{n!}\\ &\quad+\frac{(m-1)(m-2)\cdots(m-1-n+2)}{(n-1)!}\\ &= \frac{(m-1)(m-2)\cdots(m-1-n+2)}{n!}(m-1-n+1+n)\\ &= \frac{m(m-1)\cdots(m-n+1)}{n!} = \binom{m}{n}.\end{aligned}$$

Let $y = f_m(x)$. Then we obtain the differential equation

$$\begin{aligned}(1+x)\frac{dy}{dx} = my \qquad \frac{dy}{y} &= \frac{m\,dx}{1+x}\\ \Longrightarrow \int\frac{dy}{y} &= \int\frac{m\,dx}{1+x}\\ \Longrightarrow \ln|y| = m\ln|1+x| + C &= \ln|1+x|^m + k\\ \Longrightarrow |y| = e^{\ln|y|} = e^k|1+x|^m &\Longrightarrow y = C|1+x|^m,\end{aligned}$$

where $C = \pm e^k$. By putting $x = 0$,

$$C = (1+0)^m = y(0) = 1 + \sum_{n=1}^{\infty}\binom{m}{n}0^n = 1.$$

Thus $y = |1+x|^m$ or

$$(1+x)^m = 1 + \sum_{n=1}^{\infty}\binom{m}{n}x^n$$

for $|x| < 1$ because $1 + x > 0$ when $|x| < 1$. $\square$