

1. Prove the following limits by using $\epsilon - N$ definition

$$\text{i) } \lim_{n \rightarrow \infty} \frac{2n^2}{3n^2 + 2} = \frac{2}{3}.$$

$$\text{ii) } \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = 1.$$

2. For each of the following statements, either prove that the statement is true or give a counter example to show that the statement is false:

i) If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

[Hint: You may use Theorem 1.4.3.]

ii) If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n b_n\}$ diverges.

[From Question 3 onwards, you may assume the limits of the standard sequences.]

3. Evaluate the following limits:

$$\text{i) } \lim_{n \rightarrow \infty} \frac{n^2 + 5n^3 - 1}{3n^3 + 6n + 4};$$

$$\text{ii) } \lim_{n \rightarrow \infty} \frac{3^n + n^8}{2n^2 + 7^n};$$

$$\text{iii) } \lim_{n \rightarrow \infty} \sqrt{\frac{n^4 + 4n^3 + 1}{n^3 + 2n^2}}.$$

4. Use the Squeeze theorem to find the following limits:

$$\text{i) } \lim_{n \rightarrow \infty} \frac{1 + |\sin n|}{2n};$$

$$\text{ii) } \lim_{n \rightarrow \infty} \left(\frac{2n-5}{3n+1} \right)^n.$$

5. Do the following sequences tend to $+\infty$ or $-\infty$? Justify your answer.

$$\text{i) } \left\{ \frac{e^n}{n^{100}} \right\};$$

$$\text{ii) } \left\{ \frac{n}{\ln \frac{1}{n+2}} \right\}.$$

6. Evaluate the following limits (you may assume the limits of the standard sequences and use the Squeeze theorem, etc.)

- (a). $\lim_{n \rightarrow \infty} \sqrt[3]{\frac{2n^4 + n + 1}{16n^4 + n^2 + 2}}$;
- (b). $\lim_{n \rightarrow \infty} \left(3 + \ln\left(\cos \frac{1}{\sqrt{n}}\right) + \frac{n^2}{1.1^n} \right)$;
- (c). $\lim_{n \rightarrow \infty} \frac{n^4 + 8^n}{9^n + n + 8^n}$;
- (d). $\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{3}}$;
- (e). $\lim_{n \rightarrow \infty} (\sqrt{3n-2} - \sqrt{3n-3})$;
- (f). $\lim_{n \rightarrow \infty} \left(\frac{3 + (-1)^n}{5} \right)^n$;
- (g). $\lim_{n \rightarrow \infty} \frac{7^n + \ln n - n!}{n! + n^2}$;
- (h). $\lim_{n \rightarrow \infty} \frac{n^{100} 100^n}{n!}$;
- (i). $\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}}$;
- (j). $\lim_{n \rightarrow \infty} (3^n + 4^n)^{\frac{1}{n}}$;
- (k). $\lim_{n \rightarrow \infty} n \sin \frac{3}{n}$; (Hint: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.)
- (l). $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n} \right)^{2n}$;
- (m). $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^3} \right)^{n^3+2}$.

7. Let S and T be two bounded sets of real numbers. Show that $S \cup T$ is also a bounded set.

8. i) Show that a sequence $\{a_n\}$ is bounded if and only if $\{|a_n|\}$ is bounded.
 ii) Using i) or otherwise, show that if $\lim_{n \rightarrow \infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n \rightarrow \infty} a_n b_n = 0$.

Some suggested answers:

2. i) True ii) False counter-example: $a_n = 1/n$, $b_n = n$.

3. i) $\frac{5}{3}$ ii) 0 iii) $+\infty$.

4. i) 0 ii) 0

5. i) $\lim_{n \rightarrow \infty} \frac{e^n}{n^{100}} = +\infty$ ii) $\lim_{n \rightarrow \infty} \frac{n}{\ln \frac{1}{n+2}} = -\infty$.

6. (a) $1/2$ (b) 3 (c) 0 (d) 1 (e) 0 (f) 0 (g) -1 (h) 0 (i) 0 (j) 4
(k) 3 (l) $e^{-\frac{2}{3}}$ (m) e .