1. Denote the set of rational numbers by $\mathbb{Q}$. Consider the set
$$S = \{x \in \mathbb{Q} | 0 \leq x < 1\}.$$
Find $\sup S$ and $\inf S$. Justify your answers.
2. Let A and B be two non-empty bounded set of real numbers such that $A \subseteq B$. Show that $\inf A \geq \inf B$.
3. Let A and B be two non-empty bounded set of real numbers
   i) Show that $\sup A \cup B = \max\{\sup A, \sup B\}$.
   ii) Is it true that $\sup A \cap B = \min\{\sup A, \sup B\}$? Justify your answer.
4. Consider the sequence $\{a_n\}$ defined recursively by
$$a_1 = 2, \quad a_n = \sqrt{6 + a_{n-1}}, \quad n = 2, 3, 4, \cdots .$$
   i) Show that $2 \leq a_n \leq 3$ for all n.
   ii) Show that $\{a_n\}$ is monotone increasing.
   iii) Using parts i) and ii), show that $\{a_n\}$ converges, and find its limit.
5. Consider the sequence $\{x_n\}$ defined recursively by
$$x_1 = \frac{3}{4}, \quad x_{n+1} = 2x_n - x_n^2, \quad n = 1, 2, 3, \cdots .$$
Show that $\{x_n\}$ converges, and find its limit. (Hint: Show that $x_n \leq 1$ for all n and $\{x_n\}$ is monotone increasing.)
6. Find the lim sup and lim inf of the sequences:
   (a). $\{4 + \cos \frac{n\pi}{2}\}$.
   (b). $\{\frac{1+(-1)^n}{n}\}$.
7. Let $\{a_n\}$ be a bounded sequence of real numbers. Show that
$$\limsup_{n\to\infty} \sqrt{|a_n|} = \sqrt{\limsup_{n\to\infty} |a_n|}.$$

As we explained in class, roughly speaking, lim sup is the **largest** limit of convergent subsequences and lim inf is the **smallest** limit of convergent subsequences. More precisely, we have the following.

8. Let $\{a_n\}$ be any sequence. Let $B = \overline{\lim_{n\to\infty}} a_n$ and let $C = \underline{\lim}_{n\to\infty} a_n$. Suppose that B and C are finite. Show that
   (i) Let $\{a_{n_k}\}$ be any convergent subsequence of $\{a_n\}$. Then
$$C = \underline{\lim}_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \overline{\lim_{n\to\infty}} a_n = B.$$
   (ii) There exists a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that
$$\lim_{k\to\infty} a_{n_k} = B.$$
   (iii) There exists a convergent subsequence $\{a_{m_k}\}$ of $\{a_n\}$ such that
$$\lim_{k\to\infty} a_{m_k} = C.$$

Note. The statements in Question 8 still hold even if B and/or C are not finite.

Some suggested answers:

1. $\sup S = 1$ and $\inf S = 0$.
4. $\lim_{n\to\infty} a_n = 3$.
5. $\lim_{n\to\infty} x_n = 1$.
6. a) $\limsup = 5$ and $\liminf = 3$.
6. b) $\limsup = \liminf = \lim = 0$.

You should try to prove Questions 2 and 3 by yourself first. Below I give a solution of Question 4 and then you should try to solve Question 5. Also I give a solution of Question 8 (ii) and then you should try part (i) and (iii) of Question 8.

Solution to Question 4: (i) We prove that $2 \le a_n \le 3$ by induction on n. Since $a_1 = 2$, we have $2 \le a_1 \le 3$. Suppose that $a_{n-1} \le 3$ with $n \ge 2$. Then

$$2 \le \sqrt{6+2} \le a_n = \sqrt{6+a_{n-1}} \le \sqrt{6+3} = 3.$$

The induction is finished and hence the statement.

(ii) Let $n \ge 2$. Then

$$a_n - a_{n-1} = \sqrt{6+a_{n-1}} - a_{n-1} = \frac{(\sqrt{6+a_{n-1}} - a_{n-1})(\sqrt{6+a_{n-1}} + a_{n-1})}{\sqrt{6+a_{n-1}} + a_{n-1}}$$

$$= \frac{6 + a_{n-1} - a_{n-1}^2}{\sqrt{6+a_{n-1}} + a_{n-1}} \ge 0$$

because $\sqrt{6+a_{n-1}} + a_{n-1} > 0$ and $6 + x - x^2 = -(x-3)(x+2) \ge 0$ for $-2 \le x \le 3$. Thus $\{a_n\}$ is monotone increasing.

(iii) By (i) and (ii), $\{a_n\}$ is bounded above and monotone increasing. Thus $\{a_n\}$ is convergent. Let $A = \lim\limits_{n\to\infty} a_n$. Then we have the equation

$$A = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{6+a_{n-1}} = \sqrt{6 + \lim_{n\to\infty} a_{n-1}} = \sqrt{6+A}$$

and so $A^2 = 6 + A$. It follows that $A = -2$ or 3. Since $a_n \ge 2$ for each n, $A = \lim\limits_{n\to\infty} a_n \ge 2$ and so $A = 3$.

Solution to Question 8 (ii). We construct a subsequence of $\{a_n\}$ as follows. Since

$$b_1 = \sup\{a_1, a_2, \cdots\},$$

$b_1 - 1$ is not an upper bound of $\{a_1, a_2, \cdots\}$ and so there exists a_{n_1} such that

$$a_{n_1} > b_1 - 1.$$

Since

$$b_{n_1+1} = \sup\{a_{n_1+1}, a_{n_1+2}, \cdots\},$$

$b_{n_1+1} - \dfrac{1}{2}$ is not an upper bound of $\{a_{n_1+1}, a_{n_1+2}, \cdots\}$ and so there exists a_{n_2} such that $n_2 > n_1$ and

$$a_{n_2} > b_{n_1+1} - \frac{1}{2}.$$

Now, by induction, suppose that we have constructed $a_{n_1}, a_{n_2}, \cdots, a_{n_k}$ such that $n_1 < n_2 < \cdot < n_k$ and

$$a_{n_s} > b_{n_{s-1}+1} - \frac{1}{s}$$

for $1 \leq s \leq k$. Since

$$b_{n_k+1} = \sup\{a_{n_k+1}, a_{n_k+2}, \cdots\},$$

$b_{n_k+1} - \dfrac{1}{k+1}$ is not an upper bound of $\{a_{n_k+1}, a_{n_k+2}, \cdots\}$ and so there exists $a_{n_{k+1}}$ such that $n_{k+1} > n_k$ and

$$a_{n_{k+1}} > b_{n_k+1} - \frac{1}{k+1}.$$

The induction is finished and so we obtain a subsequence $\{a_{n_1}, a_{n_2}, \cdots\}$ with the property that

$$a_{n_k} > b_{n_{k-1}+1} - \frac{1}{k}$$

for any k. Consider the inequality

$$b_{n_{k-1}+1} - \frac{1}{k} < a_{n_k} \leq b_{n_k}.$$

Since $\{b_n\}$ is convergent, we have

$$\lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty} b_{n_{k-1}+1} = \lim_{n\to\infty} b_n = B$$

and

$$\lim_{k\to\infty} (b_{n_{k-1}+1} - \frac{1}{k}) = B - 0 = B.$$

Thus, by the Squeeze theorem, we have

$$\lim_{k\to\infty} a_{n_k} = B = \overline{\lim_{n\to\infty}}\, a_n.$$