NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2002-2003

MA2108 ADVANCED CALCULUS II

April 2003 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{5 + \ln\left(\cos\frac{\ln n}{\sqrt{n}}\right) + \frac{n^3}{1.1^n}\right\}.$

(b) $\left\{\frac{8^n \cdot n^{100} + \ln n - n!}{n! + n^2}\right\}.$

(c) $\left\{\left(\frac{n}{n-1}\right)^{2n+\ln n}\right\}.$

(d) $\left\{\frac{\sqrt[n]{n!}}{n}\right\}.$

Solution. (a).

$$\lim_{n\to\infty} 5 + \ln\left(\cos\frac{\ln n}{\sqrt{n}}\right) + \frac{n^3}{1.1^n} = 5 + \ln\cos 0 + 0 = 5 + 0 + 0 = 5.$$

(b).

$$\lim_{n\to\infty}\frac{8^n \cdot n^{100} + \ln n - n!}{n! + n^2} = \lim_{n\to\infty}\frac{\frac{n^{100}}{2^n}\cdot\frac{16^n}{n!} + \frac{\ln n}{n}\cdot\frac{n}{2^n}\cdot\frac{2^n}{n!} - 1}{1 + \frac{n^2}{2^n}\cdot\frac{2^n}{n!}} = -1.$$

(c).

$$\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{2n+\ln n} = \lim_{n\to\infty}\frac{1}{\left[\left(1-\frac{1}{n}\right)^n\right]^{2+\frac{\ln n}{n}}} = \frac{1}{e^{-2-0}} = e^2.$$

(d). Let $a_n = \frac{n!}{n^n}$. Then

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty}\frac{(n+1)!\cdot n^n}{(n+1)^{n+1}\cdot n!} = \lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} = e^{-1}.$$

∎

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $$\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^n}.$$

(b) $$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{1.1}}.$$

(c) $$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! \cdot 3^n}.$$

(d) $$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right).$$

Solution. (a). Divergence by the divergence test because

$$\lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0.$$

(b). Let $a_n = \frac{(\ln n)^2}{n^{1.1}}$ and let $b_n = \frac{1}{n^{1.05}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{1.1}} \cdot n^{1.05} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{0.05}} = 0,$$

that is, $a_n << b_n$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{1.05}}$ converges by p-series, the series converges by limit comparison test.

(c). Let $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!\cdot 3^n}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)\cdot n!\cdot 3^n}{(n+1)!\cdot 3^{n+1}\cdot 1\cdot 3\cdot 5\cdots(2n-1)} \\ &= \lim_{n\to\infty} \frac{(2n+1)}{(n+1)\cdot 3} = \frac{2}{3} < 1\end{aligned}$$

and so the series converges by ratio test.

(d). Let $a_n = \ln\left(1+\frac{1}{n^2}\right)$ and let $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} \overset{x=\frac{1}{n^2}}{=\!=} \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series, the series converges by the limit comparison test. ∎

Question 3 [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) $$\sum_{k=1}^{\infty} \frac{2^k}{k^2}(3x+1)^k.$$

(b) $$\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^{k^2} x^{k^2}.$$

Solution. (a). $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^2}(3x+1)^k = \sum_{k=1}^{\infty} \frac{2^k \cdot 3^k}{k^2}\left(x+\frac{1}{3}\right)^k$. Radius of convergence

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{6^n}{n^2}}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{6} = \frac{1}{6}.$$

(b). Let $a_n = \left(1-\frac{1}{n}\right)^{n^2} x^{n^2}$. Then

$$\begin{aligned} \ell = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n |x|^n \\ &= \begin{cases} +\infty & > 1 \qquad |x| > 1 \\ e^{-1} & < 1 \qquad |x| = 1 \\ 0 & < 1 \qquad |x| < 1. \end{cases} \end{aligned}$$

By root test, the series $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2} x^{n^2}$ converges if and only if $|x| \le 1$ and so the radius of convergence is 1. ∎

Question 4 [18 marks]

(a) Determine whether the following sequence of functions converge uniformly on the indicated intervals. Justify your answers.

$$F_n(x) = \frac{x^n \cos nx}{1+x^n}, \quad x \in [0, \tfrac{2}{3}].$$

(b) Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(i) $$\sum_{k=1}^{\infty} \frac{\sin kx}{k^2+x^2}, \quad x \in [0, \infty).$$

(ii) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k+3x} \quad x\in[0,\infty).$

Solution. Since, for $x\in[0,\frac{2}{3}]$,

$$-\frac{x^n}{1+x^n}\le\frac{x^n\cos nx}{1+x^n}\le\frac{x^n}{1+x^n}$$

and $\displaystyle\lim_{n\to\infty}\frac{x^n}{1+x^n}=0$, $F(x)=\displaystyle\lim_{n\to\infty}F_n(x)=0$ by the Squeeze Theorem. Observe that

$$0\le T_n=\sup_{0\le x\le\frac{2}{3}}|F_n(x)-F(x)|=\sup_{0\le x\le\frac{2}{3}}\frac{x^n|\cos nx|}{1+x^n}\le\left(\frac{2}{3}\right)^n.$$

Since $\displaystyle\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0$, $\displaystyle\lim_{n\to\infty}T_n=0$ by the Squeeze Theorem and so the sequence of functions converges uniformly on $[0,\frac{2}{3}]$ by the T-test.

(b)(i) Since

$$\left|\frac{\sin nx}{n^2+x^2}\right|\le\frac{1}{n^2}$$

and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by p-series, the series of functions converges uniformly on $[0,\infty)$ by Weierstrass M-test.

(b)(ii) Let $S(x)=\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k+3x}$ and $S_n(x)=\displaystyle\sum_{k=1}^{n}\frac{(-1)^k}{k+3x}$. For each $x\in[0,\infty)$, the sequence $\left\{\frac{1}{k+3x}\right\}$ is positive monotone decreasing with $\displaystyle\lim_{k\to\infty}\frac{1}{k+3x}=0$. By Alternating series estimation,

$$|S_n(x)-S(x)|\le\frac{1}{n+1+3x}\le\frac{1}{n+1}$$

for each n and each $x\ge 0$. Thus

$$0\le T_n=\sup_{x\ge 0}|S_n(x)-S(x)|\le\frac{1}{n+1}.$$

Since $\displaystyle\lim_{n\to\infty}\frac{1}{n+1}=0$, $\displaystyle\lim_{n\to\infty}T_n=0$ and so the series of functions converges uniformly on $[0,\infty)$.

∎

SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Evaluate $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{n+\cos(nx^2)}{x^2+n}\,dx$. Justify your answer.

(b) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{2^n}{n}(2x+1)^n.$$

Justify your answer.

(c) Suppose that $\displaystyle f(x)=\sum_{k=0}^{\infty}c_kx^k$ and $\displaystyle e^{f(x)}=\sum_{k=0}^{\infty}d_kx^k.$ has positive radius of convergence. Show that, for each $n\geq 1$,

$$d_n=\frac{1}{n}\sum_{k=1}^{n}kc_kd_{n-k}.$$

Solution. (a). Let $F_n(x)=\frac{n+\cos(nx^2)}{x^2+n}$. Then

$$F(x)=\lim_{n\to\infty}F_n(x)=\lim_{n\to\infty}\frac{n+\cos(nx^2)}{x^2+n}=\lim_{n\to\infty}\frac{1+\frac{\cos(nx^2)}{n}}{\frac{x^2}{n}+1}=\frac{1+0}{0+1}=1$$

and

$$0\leq T_n=\sup_{0\leq x\leq 1}|F_n(x)-F(x)|=\sup_{0\leq x\leq 1}\left|\frac{n+\cos(nx^2)}{x^2+n}-1\right|$$

$$=\sup_{0\leq x\leq 1}\left|\frac{\cos(nx^2)-x^2}{x^2+n}\right|\leq\frac{2}{n}.$$

Since $\displaystyle\lim_{n\to\infty}\frac{2}{n}=0$, $\displaystyle\lim_{n\to\infty}T_n=0$ by Squeeze Theorem and so the $F_n(x)$ converges uniformly to $F(x)$ on $[0,1]$. Thus

$$\lim_{n\to\infty}\int_0^1\frac{n+\cos(nx^2)}{x^2+n}\,dx=\int_0^1\lim_{n\to\infty}\frac{n+\cos(nx^2)}{x^2+n}\,dx=\int_0^1 1\,dx=1.$$

(b). $\sum_{n=1}^{\infty} \frac{2^n}{n}(2x+1)^n = \sum_{n=1}^{\infty} \frac{4^n}{n}\left(x+\frac{1}{2}\right)^n$. The radius of convergence

$$R = \frac{1}{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim_{n\to\infty} \frac{4^{n+1}\cdot n}{(n+1)\cdot 4^n}} = \frac{1}{\lim_{n\to\infty} \frac{4}{1+\frac{1}{n}}} = \frac{1}{4}.$$

Check the ending points $x_0 \pm R = -\frac{1}{2} \pm \frac{1}{4}$. When $x = -\frac{1}{2}+\frac{1}{4}$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which is divergent by p-series. When $x = -\frac{1}{2}-\frac{1}{4}$, the series is

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

which is convergent by the alternating series test because $\frac{1}{n}$ is monotone decreasing and tends to 0. Thus the interval of convergence is $[-\frac{3}{4}, \frac{1}{4})$.

(c). Let R_1 and R_2 denote radius of convergence of $\sum_{k=0}^{\infty} c_k x^k$ and $\sum_{k=0}^{\infty} d_k x^k$, respectively. Then R_1 and R_2 are positive by the assumption. Let $R = \min\{R_1, R_2\}$. Then, on $(-R, R)$, the power series

$$\begin{aligned}\sum_{n=1}^{\infty} n d_n x^n &= x \cdot \left(e^{f(x)}\right)' = x \cdot e^{f(x)} \cdot f'(x) = e^{f(x)} \cdot x \cdot f'(x)\\ &= \left(\sum_{k=0}^{\infty} d_k x^k\right) \cdot \left(\sum_{k=1}^{\infty} k c_k x^k\right)\\ &= \sum_{n=1}^{\infty} (n c_n d_0 + (n-1)c_{n-1} d_1 + \cdots + c_1 d_{n-1}) x^n.\end{aligned}$$

By the uniqueness of power series,

$$n d_n = \sum_{k=1}^{n} k c_k d_{n-k}$$

and so

$$d_n = \frac{1}{n} \sum_{k=1}^{n} k c_k d_{n-k}.$$

∎

Question 6 [20 marks]

(a) Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n} \cdot x}{e^{n^2 x}}.$$

Is $f(x)$ continuous on $[0, +\infty)$? Justify your answer.

(b) Using any applicable method, find the Taylor series of the function $f(x) = \ln\left(\frac{1+2x}{1-x}\right)$ at $x_0 = 0$, and specify the interval on which the series converges to the function.

(c) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$. Prove that

$$\varliminf_{n\to\infty} a_n + \varliminf_{n\to\infty} b_n \le \varliminf_{n\to\infty} (a_n + b_n).$$

Solution. (a). Let $f_n(x) = \frac{\sqrt{n}\cdot x}{e^{n^2 x}}$. Then

$$f_n'(x) = \frac{\sqrt{n}e^{n^2x} - \sqrt{n}x \cdot e^{n^2x} \cdot n^2}{e^{2n^2x}} = \frac{\sqrt{n}(1-n^2x)}{e^{n^2x}}.$$

Since $f_n'(x) \ge 0$ for $0 \le x \le \frac{1}{n^2}$ and $f'(x) \le 0$ for $x \ge \frac{1}{n^2}$, the maximum of $f_n(x)$ on $[0, \infty)$ is

$$f_n\left(\frac{1}{n^2}\right) = \frac{\sqrt{n}\cdot\frac{1}{n^2}}{e} = \frac{1}{en^{3/2}}$$

that is

$$|f_n(x)| \le \frac{1}{en^{3/2}}$$

for all $x \in [0, \infty)$. Since $\sum_{n=1}^{\infty} \frac{1}{en^{3/2}} = \frac{1}{e}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly by Weierstrass M-test. Because each $f_n(x)$ is continuous on $[0, \infty)$, the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n} \cdot x}{e^{n^2 x}}.$$

is continuous on $[0, \infty)$.

(b).

$$f(x) = \ln(1+2x) - \ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-x)^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{\left((-1)^{n+1}\cdot 2^n + 1\right)x^n}{n}.$$

To work out the interval on which the series converges to the function. Observe that the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2x)^n}{n}$$

converges to $\ln(1+2x)$ when $-1<2x\le 1$, or $-\frac{1}{2}<x\le\frac{1}{2}$. And the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-x)^n}{n}$$

converges to $\ln(1-x)$ when $-1<-x\le 1$ or $-1\le x<1$. Hence the interval of the power series

$$\sum_{n=1}^{\infty}\frac{((-1)^{n+1}\cdot 2^n+1)\,x^n}{n}$$

converging to $f(x)$ is $(-\frac{1}{2},\frac{1}{2}]\cap[-1,1)=(-\frac{1}{2},\frac{1}{2}]$.

(c). Let

$$\begin{aligned} c_n &= \inf\{a_n, a_{n+1}, \ldots\},\\ d_n &= \inf\{b_n, b_{n+1}, \ldots\},\\ e_n &= \inf\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\}. \end{aligned}$$

Then

$$c_n\le a_k$$

for all $k\ge n$ and

$$d_n\le b_k$$

for all $k\ge n$. For each $k\ge n$,

$$c_n+d_n\le a_k+b_k$$

and so c_n+d_n is a lower bound of $\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\}$. Hence

$$c_n+d_n\le e_n=\inf\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\}$$

because e_n is the greatest lower bound. By definition,

$$\varliminf_{n\to\infty} a_n+\varliminf_{n\to\infty} b_n=\lim_{n\to\infty} c_n+\lim_{n\to\infty} d_n=\lim_{n\to\infty}(c_n+d_n)\le\lim_{n\to\infty} e_n=\varliminf_{n\to\infty}(a_n+b_n).$$

This finishes the proof. ∎

Question 7 [20 marks]

(a) Let $f(x) = x^2 \cdot \sqrt[3]{1+x^9}$. Find $f^{(28)}(0)$.

(b) Determine the absolute convergence, conditional convergence or divergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n)}{n(\ln n+1)\left[\ln(\ln n+1)\right]^2}$. Justify your answers.

(c) Does the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^x}$ converge uniformly on $(1,+\infty)$? Justify your answer.

Solution. (a).

$$f(x) = x^2\cdot\sqrt[3]{1+x^9} = x^2\cdot\left(1+\sum_{k=1}^{\infty}\binom{\frac{1}{3}}{k}\left(x^9\right)^k\right) = x\cdot\left(1+\sum_{k=1}^{\infty}\binom{\frac{1}{3}}{k}x^{9k}\right)$$

$$= x^2\cdot\left(1+\frac{1}{3}x^9+\binom{\frac{1}{3}}{2}x^{18}+\binom{\frac{1}{3}}{3}x^{27}+\binom{\frac{1}{3}}{4}x^{36}+\cdots\right)$$

$$= x^2+\frac{1}{3}x^{11}+\binom{\frac{1}{3}}{2}x^{20}+\binom{\frac{1}{3}}{3}x^{29}+\cdots.$$

Thus

$$\frac{f^{(28)}(0)}{(28)!} = 0$$

and $f^{(28)}(0) = 0$.

(b). Note that

$$\left|\frac{\cos(n)}{n(\ln n+1)\left[\ln(\ln n+1)\right]^2}\right| \le \frac{1}{n(\ln n+1)\left[\ln(\ln n+1)\right]^2}.$$

Let

$$f(x) = \frac{1}{x(\ln x+1)\left[\ln(\ln x+1)\right]^2}.$$

Then $f(x)$ is positive monotone decreasing on $[2,+\infty)$. Since

$$\int_2^{\infty}\frac{1}{x(\ln x+1)\left[\ln(\ln x+1)\right]^2}\,dx \underset{dy=\frac{dx}{x}}{\overset{y=\ln x+1}{=\!=\!=\!=}} \int_{\ln 2+1}^{\infty}\frac{1}{y(\ln y)^2}\,dy$$

$$\underset{dz=\frac{dy}{y}}{\overset{z=\ln y}{=\!=\!=\!=}} \int_{\ln(\ln 2+1)}^{\infty}\frac{1}{z^2}\,dz = -\frac{1}{z}\bigg|_{\ln(\ln 2+1)}^{\infty} = \frac{1}{\ln(\ln 2+1)}.$$

Thus the series

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n+1)\left[\ln(\ln n+1)\right]^2}$$

converges by integral test and, by comparison test, the series

$$\sum_{n=2}^{\infty}\left|\frac{\cos n}{n(\ln n+1)\,[\ln(\ln n+1)]^2}\right|$$

converges. Hence the given series is absolutely convergent.

(c). NO. Prove by contradiction. Suppose that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^x}$ converge uniformly on $(1,+\infty)$. Let $S_n(x)=\displaystyle\sum_{k=1}^{n}\frac{1}{(2n-1)^x}$. By Cauchy Criterion, given any $0<\epsilon<\frac{1}{4}$, there exists N such that

$$|S_m(x)-S_n(x)|=\left|\sum_{k=n+1}^{m}\frac{1}{(2k-1)^x}\right|<\epsilon$$

for all $m>n>N$ and for all $x>1$. In particular, for all $n>N$,

$$|S_{2n}(x)-S_n(x)|=\left|\frac{1}{(2n+1)^x}+\frac{1}{(2n+3)^x}+\cdots+\frac{1}{(4n-1)^x}\right|<\epsilon$$

for all $x>1$. Let x tends to 1. For $n>N$,

$$\begin{aligned}&\left|\frac{1}{(2n+1)}+\frac{1}{(2n+3)}+\cdots+\frac{1}{(4n-1)}\right|\\ &=\lim_{x\to1}\left|\frac{1}{(2n+1)^x}+\frac{1}{(2n+3)^x}+\cdots+\frac{1}{(4n-1)^x}\right|<\epsilon.\end{aligned}$$

Observe that

$$\left|\frac{1}{(2n+1)}+\frac{1}{(2n+3)}+\cdots+\frac{1}{(4n-1)}\right|\geq n\cdot\frac{1}{(4n-1)}\geq n\cdot\frac{1}{4n}=\frac{1}{4}$$

for all n. It follows that for each $n>N$,

$$\frac{1}{4}\leq\left|\frac{1}{(2n+1)}+\frac{1}{(2n+3)}+\cdots+\frac{1}{(4n-1)}\right|<\epsilon.$$

This contradicts to that $\epsilon<\frac{1}{4}$ and hence the result. ■