Question 1. [2 points, 1 for each part]

Prove the following limits by using \( \epsilon - N \) definition

\[ \lim_{n \to \infty} \frac{3n + 8}{2n + 9} = \frac{3}{2}. \]
\[ \lim_{n \to \infty} \frac{\frac{(-1)^n}{n^2 + 1}}{n} = 0. \]

Proof. (i). Note that

\[
\left| \frac{3n + 8}{2n + 9} - \frac{3}{2} \right| = \left| \frac{2(3n + 8) - 3(2n + 9)}{2(2n + 9)} \right| = \frac{11}{2(2n + 9)} < \frac{11}{4n} < \frac{3}{n}. 
\]

Given \( \epsilon > 0 \), choose \( N \) such that \( \frac{3}{N} \leq \epsilon \iff N \geq \frac{3}{\epsilon} \). When \( n > N \),

\[
\frac{3n + 8}{2n + 9} - \frac{3}{2} < \frac{3}{n} < \frac{3}{N} \leq \epsilon.
\]

(ii). Note that

\[
\left| \frac{(-1)^n}{n^2 + 1} - 0 \right| = \frac{n}{n^2 + 1} < \frac{n}{n^2} = \frac{1}{n}.
\]

Given \( \epsilon > 0 \), choose \( N \) such that \( \frac{1}{N} \leq \epsilon \iff N \geq \frac{1}{\epsilon} \). When \( n > N \),

\[
\left| \frac{(-1)^n}{n^2 + 1} - 0 \right| < \frac{1}{n} < \frac{1}{N} \leq \epsilon.
\]

\qed

Question 2. [5 points, 1 for each part]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) \( \left\{ \left( \sqrt{n^2 + n} - n \right) \right\} \).

(b) \( \left\{ \left( 2^n + 3^n \right)^{\frac{1}{n}} \right\} \).

(c) \( \left\{ \frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n} \right\} \).

(d) \( \left\{ \left( \frac{3n}{3n-1} \right)^{2n+\sqrt{n}} \right\} \).

(e) \( \left\{ \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \right\} \).
Solution. (a).

\[
\lim_{n \to \infty} \left( \sqrt{n^2 + n} - n \right) = \lim_{n \to \infty} \frac{(\sqrt{n^2 + n} - n) \cdot (\sqrt{n^2 + n} + n)}{\sqrt{n^2 + n} + n} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + n} + n} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n} + 1} = \frac{1}{2}.
\]

(b).

\[
\lim_{n \to \infty} (2^n + 3^n)^{\frac{1}{n}} = \lim_{n \to \infty} 3 \left[ \left( \frac{2}{3} \right)^n + 1 \right]^{\frac{1}{n}} = 3 \cdot (0 + 1)^0 = 3.
\]

Another solution: Since

\[
3 = (3^n)^{\frac{1}{n}} \leq (2^n + 3^n)^{\frac{1}{n}} \leq (3^n + (3^n)^{\frac{1}{n}})^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot 3 \quad \text{and} \quad \lim_{n \to \infty} 2^{\frac{1}{n}} \cdot 3 = 3,
\]

\[
\lim_{n \to \infty} (2^n + 3^n)^{\frac{1}{n}} = 3 \quad \text{by the Squeeze Theorem.}
\]

(c).

\[
\lim_{n \to \infty} \sqrt{n! + 2n^5 + \ln n} \cdot \frac{n}{n! + 5n + 3n} = \lim_{n \to \infty} \sqrt{\frac{1 + 2n^5 + \ln n}{n! + 5n + 3n}} = \sqrt{\frac{1 + 0 + 0}{1 + 0 + 0}} = 1.
\]

(d).

\[
\lim_{n \to \infty} \left( \frac{3n}{3n - 1} \right)^{2n + \sqrt{n}} = \lim_{n \to \infty} \frac{1}{\left( \frac{3n - 1}{3n} \right)^{2n + \sqrt{n}}} = \lim_{n \to \infty} \frac{1}{\left[ \left( \frac{1 - \frac{1}{3n}^{\frac{3n}{n}} \cdot \frac{3}{3} \right)^{\frac{3n}{n}} \right]^3} = \left( \frac{1 - e^{-\frac{1}{2}}} {e^{-\frac{1}{2}}} \right)^{\frac{3n}{n}} = e^{\frac{2}{3}}.
\]

(e). Note that

\[
-\frac{n^{50} \cdot 50^n}{n!} \leq \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} \leq \frac{n^{50} \cdot 50^n}{n!}.
\]

Since

\[
\lim_{n \to \infty} \frac{n^{50} \cdot 50^n}{n!} = \lim_{n \to \infty} \frac{n^{50} \cdot 2^n}{n!} = \lim_{n \to \infty} \frac{n^{50} 10^n}{2^n n!} = 0 \cdot 0 = 0\]

\[
\lim_{n \to \infty} \frac{n^{50} \cdot 50^n \cdot \sin n}{n!} = 0 \quad \text{by the Squeeze Theorem.}
\]

Question 3. [3 points, 1 for each part]

(a) If \( \{ a_n \} \) is convergent, show that \( \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n \).

(b) A sequence \( \{ a_n \} \) is defined by \( a_1 = 1 \) and \( a_{n+1} = 1/(1 + a_n) \) for \( n \geq 1 \).

Assume that \( \{ a_n \} \) is convergent, find its limit.
(c) Find the limit of the sequence
\[
\left\{ \sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2}, \ldots \right\}.
\]

Solution. (a). Let \( A = \lim_{n \to \infty} a_n \). Given any \( \epsilon > 0 \), there exists \( N \) such that
\[
|a_n - A| < \epsilon
\]
for all \( n > N \) (by \( \epsilon - N \) definition). Write \( b_n \) for \( a_{n+1} \), that is \( b_n = a_{n+1} \). For all \( n > N \),
\[
|b_n - A| = |a_{n+1} - A| < \epsilon
\]
because \( n + 1 > n > N \). From \( \epsilon - N \) definition, \( \lim_{n \to \infty} b_n = A = \lim_{n \to \infty} a_n \), that is,
\[
\lim_{n \to \infty} a_{n+1} = A = \lim_{n \to \infty} a_n.
\]

(b). Let \( A = \lim_{n \to \infty} a_n \). Then
\[
A = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{1 + a_n} = \frac{1}{1 + \lim_{n \to \infty} a_n} = \frac{1}{1 + A}.
\]
Thus \( A(1 + A) = 1 \) or \( A^2 + A - 1 = 0 \). It follows that
\[
A = \frac{-1 \pm \sqrt{5}}{2}.
\]
Next we show that \( a_n > 0 \) by induction. When \( n = 1 \), \( a_1 = 1 > 0 \). Suppose that \( a_n > 0 \). Then \( a_{n+1} = 1/(1 + a_n) > 0 \). The induction is finished and so \( a_n > 0 \) for all \( n \). It follows that \( A = \lim_{n \to \infty} a_n \geq 0 \). The value \( \frac{-1 - \sqrt{5}}{2} \) is rejected because \( A \geq 0 \), and so
\[
A = \frac{-1 + \sqrt{5}}{2}.
\]

(c). From the sequence, we see that \( a_1 = \sqrt{2} \) and \( a_{n+1} = \sqrt{2}a_n \). We show that \( \lim_{n \to \infty} a_n \) exists:

First we prove by induction that \( 0 \leq a_n \leq 2 \). When \( n = 1 \), \( 0 \leq a_1 \leq 2 \) holds. Suppose that \( 0 \leq a_n \leq 2 \). Then
\[
0 \leq \sqrt{2}a_n = a_{n+1} \leq \sqrt{2} \cdot 2 = 2.
\]
The induction is finished and so \( 0 \leq a_n \leq 2 \) for all \( n \).

Next since \( 0 \leq a_n \leq 2 \), \( a_{n+1} = \sqrt{2}a_n \geq \sqrt{a_n \cdot a_n} = a_n \) for all \( n \). Thus \( \{a_n\} \) is monotone increasing. By monotone convergence theorem, \( \lim_{n \to \infty} a_n \) exists.

Finally let \( A = \lim_{n \to \infty} a_n \). Then
\[
A = \lim_{n \to \infty} a_{n+1} = \sqrt{2} \lim_{n \to \infty} a_n = \sqrt{2}A
\]
and so \( A^2 = 2A \) or \( A = 0,2 \). Since \( \{a_n\} \) is monotone increasing, \( a_n \geq a_1 = \sqrt{2} \) for all \( n \). It follows that \( A = \lim_{n \to \infty} a_n \geq \sqrt{2} \). Thus 0 is rejected and so \( A = 2 \). □